Unit II Electrostatics-II

In this chapter we will discuss on the followings:

- Electrostatic Potential, Equipotential Surfaces
- Boundary Conditions for Static Electric Fields
- Capacitance and Capacitors
- Electrostatic Energy
- Laplace's and Poisson's Equations
- Uniqueness of Electrostatic Solutions
- Method of Images
- Solution of Boundary Value Problems in Different Coordinate Systems

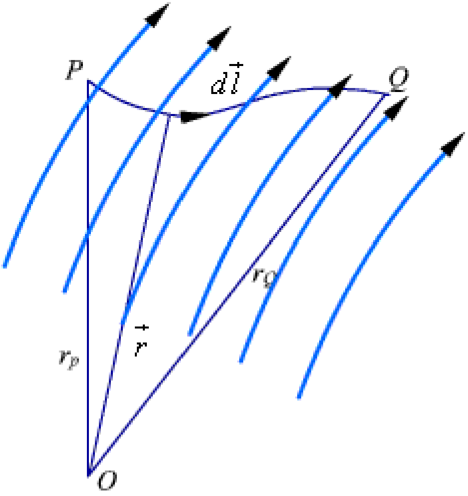
Electrostatic Potential and Equipotential Surfaces

In the previous sections we have seen how the electric field intensity due to a charge or a charge distribution can be found using Coulomb's law or Gauss's law. Since a charge placed in the vicinity of another charge (or in other words in the field of other charge) experiences a force, the movement of the charge represents energy exchange. Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field.

Let us suppose that we wish to move a positive test charge Δq from a point P to another point Q as shown in the Fig. 2.8.

The force at any point along its path would cause the particle to accelerate and move it out of the region if unconstrained. Since we are dealing with an electrostatic case, a force equal to the negative of that acting on the charge is to be applied while Δq moves from P to Q. The work done by this external agent in moving the charge by a distance $d\vec{l}$ is given by:

$$dW = -\Delta q \vec{E}.d\vec{l} \qquad \text{.............................}(2.23)$$



$$W = -\Delta q \int_P^Q \vec{E} \cdot d\vec{l} \qquad \text{.....................................}(2.24)$$

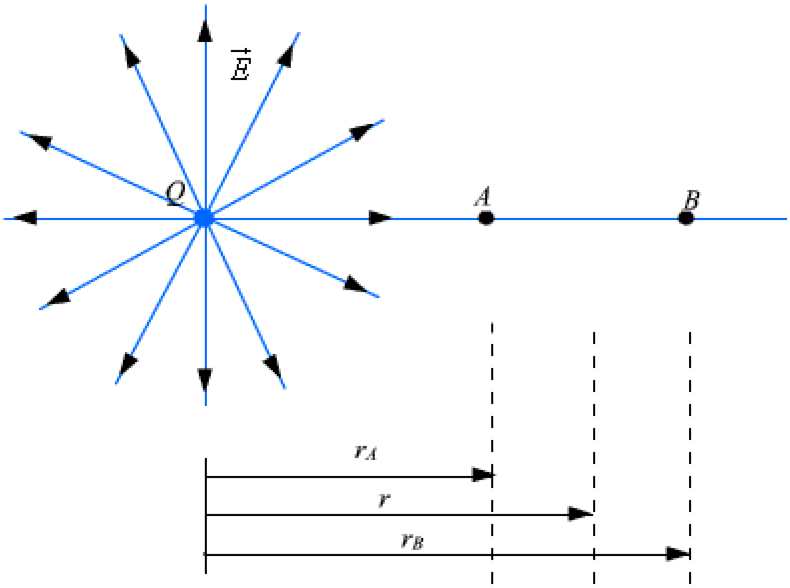
The potential difference between two points P and Q , V_{PQ}, is defined as the work done per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_P^Q \vec{E} \cdot d\vec{l} \qquad \text{...............................}(2.25)$$

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in Joules/Coulomb which is referred to as **Volts**.

Let us consider a point charge Q as shown in the Fig. 2.9.



$$V_{BA} = -\int_{B}^{A} \vec{E}\cdot d\vec{l} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B \qquad \text{..................................}(2.26)$$

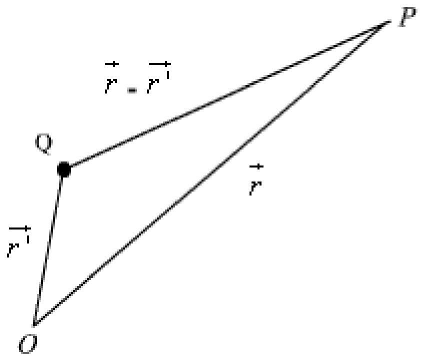
It is customary to choose the potential to be zero at infinity. Thus potential at any point (r_A = r) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. r_B = 0).

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \qquad \text{..................................}(2.27)$$

Or, in other words,

$$V = -\int_{\infty}^{r} E.dl \qquad \text{..................................(2.28)}$$

Let us now consider a situation where the point charge Q is not located at the origin as shown in Fig. 2.10.



$$V(r) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\left|\vec{r} - \vec{r}\,'\right|} \qquad \text{..................................}(2.29)$$

So far we have considered the potential due to point charges only. As any other type of charge distribution can be considered to be consisting of point charges, the same basic ideas now can be extended to other types of charge distribution also.

Let us first consider *N* point charges Q_1, Q_2,.....Q_N located at points with position vectors $\vec{r}_1$, $\vec{r}_2$,...... $\vec{r}_N$. The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{\left|\vec{r} - \vec{r}_1\right|} + \frac{Q_2}{\left|\vec{r} - \vec{r}_2\right|} + \ldots\ldots \frac{Q_N}{\left|\vec{r} - \vec{r}_N\right|}\right) \qquad \text{..................................}(2.30a)$$

or,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=n}^{N} \frac{Q_n}{\left|\vec{r} - \vec{r}_n\right|} \qquad \text{..}(2.30b)$$

For continuous charge distribution, we replace point charges Q_n by corresponding charge elements $\rho_L dl$ or $\rho_S ds$ or $\rho_V dv$ depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral. With these modifications we can write:

For line charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}')dl'}{|\vec{r}-\vec{r}_n|}$$ ....(2.31)

For surface charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\vec{r}')ds'}{|\vec{r}-\vec{r}_n|}$$ ....(2.32)

For volume charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_V(\vec{r}')dv'}{|\vec{r}-\vec{r}_n|}$$ ....(2.33)

It may be noted here that the primed coordinates represent the source coordinates and the unprimed coordinates represent field point.

Further, in our discussion so far we have used the reference or zero potential at infinity. If any other point is chosen as reference, we can write:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$ ....(2.34)

where C is a constant. In the same manner when potential is computed from a known electric field we can write:

$$V = -\int \vec{E}\cdot d\vec{l} + C$$ ....(2.35)

The potential difference is however independent of the choice of reference.

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l} = \frac{W}{Q}$$ ....(2.36)

We have mentioned that electrostatic field is a conservative field; the work done in moving a charge from one point to the other is independent of the path. Let us consider moving a charge from point P_1 to P_2 in one path and then from point P_2 back to P_1 over a different path. If the work done on the two paths were different, a net positive or negative amount of work would have been done when the body returns to its original position P_1. In a conservative field there is no mechanism for dissipating energy corresponding to any positive work neither any source is present from which energy could be absorbed in the case of negative work. Hence the question of different works in two paths is untenable, the work must have to be independent of path and depends on the initial and final positions.

Since the potential difference is independent of the paths taken, $V_{AB} = -V_{BA}$, and over a closed path,

$$V_{BA} + V_{AB} = \oint \vec{E}\cdot d\vec{l} = 0$$ ....(2.37)

Applying Stokes's theorem, we can write:

$$\oint \vec{E}\cdot d\vec{l} = \int_s \left(\nabla \times \vec{E}\right)\cdot d\vec{s} = 0 \quad \text{............................}(2.38)$$

from which it follows that for electrostatic field,

$$\nabla \times \vec{E} = 0 \quad \text{.......................................}(2.39)$$

Any vector field $\vec{A}$ that satisfies $\nabla \times \vec{A} = 0$ is called an irrotational field.

From our definition of potential, we can write

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial x}dz = -\vec{E}\cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)\cdot\left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right) = -\vec{E}\cdot d\vec{l}$$

$$\nabla V\cdot d\vec{l} = -\vec{E}\cdot d\vec{l} \quad \text{................................}(2.40)$$

from which we obtain,

$$\vec{E} = -\nabla V \quad \text{...}(2.41)$$

From the foregoing discussions we observe that the electric field strength at any point is the negative of the potential gradient at any point, negative sign shows that $\vec{E}$ is directed from higher to lower values of $\vec{V}$. This gives us another method of computing the electric field, i. e. if we know the potential function, the electric field may be computed. We may note here that that one scalar function $\vec{V}$ contain all the information that three components of $\vec{E}$ carry, the same is possible because of the fact that three components of $\vec{E}$ are interrelated by the relation $\nabla \times \vec{E}$.

Example: Electric Dipole

An electric dipole consists of two point charges of equal magnitude but of opposite sign and separated by a small distance.

As shown in figure 2.11, the dipole is formed by the two point charges Q and -Q separated by a distance d , the charges being placed symmetrically about the origin. Let us consider a point P at a distance r, where we are interested to find the field.

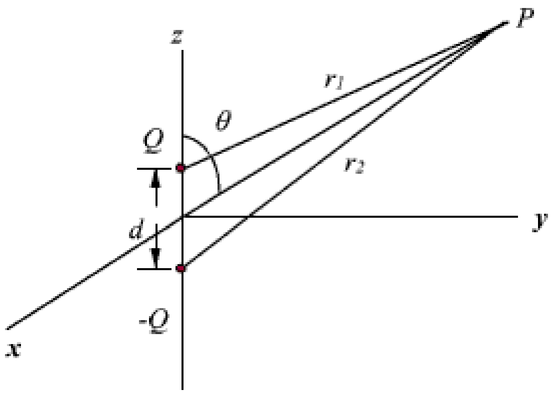


Fig *2*.11 : **Electric Dipole**

The potential at P due to the dipole can be written as:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_1} - \frac{Q}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \quad \text{..........................}(2.42)$$

When r_1 and r_2>>d, we can write $r_2 - r_1 = 2\times\frac{d}{2}\cos\theta = d\cos\theta$ and $r_1 \cong r_2 \cong r$.

Therefore,

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} \quad \text{..}(2.43)$$

We can write,

$$Qd\cos\theta = Qd\hat{a}_z\cdot\hat{a}_r \quad \text{..}(2.44)$$

The quantity $\vec{P} = Q\vec{d}$ is called the **dipole moment** of the electric dipole.

Hence the expression for the electric potential can now be written as:

$$V = \frac{\vec{P}\cdot\hat{a}_r}{4\pi\varepsilon_0 r^2} \quad \text{................................}(2.45)$$

It may be noted that while potential of an isolated charge varies with distance as $1/r$ that of an electric dipole varies as $1/r^2$ with distance.

If the dipole is not centered at the origin, but the dipole center lies at $\vec{r}\,'$, the expression for the potential can be written as:

$$V = \frac{\vec{P}\cdot(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}\,'\right|^3} \qquad \text{........................(2.46)}$$

The electric field for the dipole centered at the origin can be computed as

$$\begin{aligned}\vec{E} = -\nabla V &= -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta\right] \\ &= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{a}_\theta \\ &= \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)\end{aligned}$$

$$\vec{E} = \frac{\vec{P}}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right) \qquad \text{........................(2.47)}$$

$\vec{P} = Q\vec{d}$ is the magnitude of the dipole moment. Once again we note that the electric field of electric dipole varies as $1/r^3$ where as that of a point charge varies as $1/r^2$.

Equipotential Surfaces

An equipotential surface refers to a surface where the potential is constant. The intersection of an equipotential surface with an plane surface results into a path called an equipotential line. No work is done in moving a charge from one point to the other along an equipotential line or surface.

In figure 2.12, the dashes lines show the equipotential lines for a positive point charge. By symmetry, the equipotential surfaces are spherical surfaces and the equipotential lines are circles. The solid lines show the flux lines or electric lines of force.

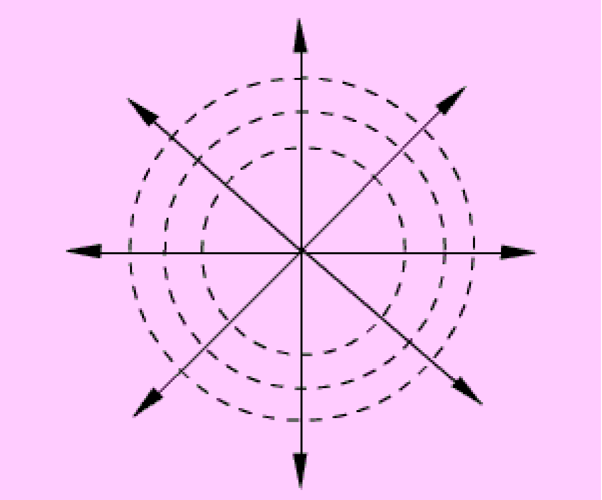


Fig *2*.12: Equipotential Lines for a Positive Point Charge

Michael Faraday as a way of visualizing electric fields introduced flux lines. It may be seen that the electric flux lines and the equipotential lines are normal to each other.

In order to plot the equipotential lines for an electric dipole, we observe that for a given Q and d, a constant V requires that $\frac{\cos\theta}{r^2}$ is a constant. From this we can write $r = c_v\sqrt{\cos\theta}$ to be the equation for an equipotential surface and a family of surfaces can be generated for various values of c_v.When plotted in 2-D this would give equipotential lines.

To determine the equation for the electric field lines, we note that field lines represent the direction of $\vec{E}$ in space. Therefore,

$\vec{dl} = k\vec{E}$, k is a constant ....*(2.48)*

$\hat{a}_r dr + r d\theta \hat{a}_\theta + \hat{a}_\phi r \sin\theta = k(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) = \vec{dl}$ ....*(2.49)*

For the dipole under consideration E_ϕ=0 , and therefore we can write,

$$\frac{dr}{E_r} = \frac{r d\theta}{E_\theta}$$

$$\frac{dr}{r} = \frac{2\cos\theta d\theta}{\sin\theta} = \frac{2d(\sin\theta)}{\sin\theta}$$ ....*(2.50)*

$$r = c_e \sin^2\theta$$

Boundary conditions for Electrostatic fields

In our discussions so far we have considered the existence of electric field in the homogeneous medium. Practical electromagnetic problems often involve media with different physical properties. Determination of electric field for such problems requires the knowledge of the relations of field quantities at an interface between two media. The conditions that the fields must satisfy at the interface of two different media are referred to as ***boundary conditions*** .

In order to discuss the boundary conditions, we first consider the field behavior in some common material media.

In general, based on the electric properties, materials can be classified into three categories: conductors, semiconductors and insulators (dielectrics). In *conductor* , electrons in the outermost shells of the atoms are very loosely held and they migrate easily from one atom to the other. Most metals belong to this group. The electrons in the atoms of *insulators* or *dielectrics* remain confined to their orbits and under normal circumstances they are not liberated under the influence of an externally applied field. The electrical properties of *semiconductors* fall between those of conductors and insulators since semiconductors have very few numbers of free charges.

The parameter *conductivity* is used characterizes the macroscopic electrical property of a material medium. The notion of conductivity is more important in dealing with the current flow and hence the same will be considered in detail later on.

If some free charge is introduced inside a conductor, the charges will experience a force due to mutual repulsion and owing to the fact that they are free to move, the charges will appear on the surface. The charges will redistribute themselves in such a manner that the field within the conductor is zero. Therefore, under steady condition, inside a conductor $\rho_v = 0$.

From Gauss's theorem it follows that

$$\vec{E} = 0 \quad \text{.......................}(2.51)$$

The surface charge distribution on a conductor depends on the shape of the conductor. The charges on the surface of the conductor will not be in equilibrium if there is a tangential component of the electric field is present, which would produce movement of the charges. Hence under static field conditions, tangential component of the electric field on the conductor surface is zero. The electric field on the surface of the conductor is normal everywhere to the surface . Since the tangential component of electric field is zero, the conductor surface is an equipotential surface. As $\vec{E}$ = 0 inside the conductor, the conductor as a whole has the same potential. We may further note that charges require a finite time to redistribute in a conductor. However, this time is very small $\sim 10^{-19}$ sec for good conductor like copper.

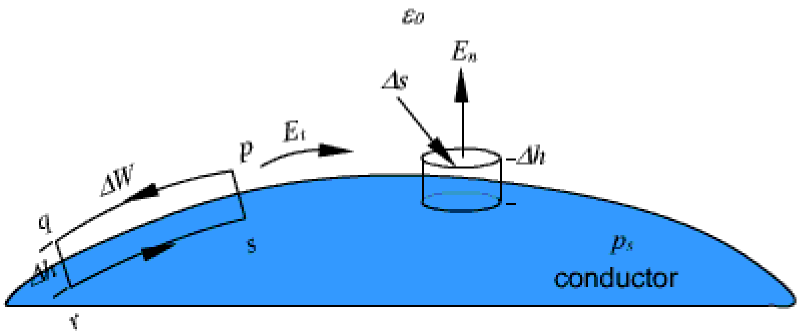


Fig 2.14: Boundary Conditions for at the surface of a Conductor

Let us now consider an interface between a conductor and free space as shown in the figure 2.14.

Let us consider the closed path *pqrsp* for which we can write,

$$\oint \vec{E}\cdot d\vec{l} = 0 \quad \text{.................................}(2.52)$$

For $\Delta h \rightarrow 0$ and noting that $\vec{E}$ inside the conductor is zero, we can write

$$E_t \Delta w = 0 \text{......................................}(2.53)$$

E_t is the tangential component of the field. Therefore we find that

$$E_t = 0 \text{ ....}(2.54)$$

In order to determine the normal component E_n, the normal component of $\vec{E}$, at the surface of the conductor, we consider a small cylindrical Gaussian surface as shown in the Fig.12. Let Δs represent the area of the top and bottom faces and Δh represents the height of the cylinder. Once again, as $\Delta h \rightarrow 0$, we approach the surface of the conductor. Since $\vec{E} = 0$ inside the conductor is zero,

$$\varepsilon_0 \oint_s \vec{E}\cdot d\vec{s} = \varepsilon_0 E_n \Delta s = \rho_s \Delta s \quad \text{.............}(2.55)$$

$$E_n = \frac{\rho_s}{\varepsilon_0} \quad \text{..................}(2.56)$$

Therefore, we can summarize the boundary conditions at the surface of a conductor as:

$$E_t = 0 \text{ ....}(2.57)$$

$$E_n = \frac{\rho_s}{\varepsilon_0} \quad \text{.....................(2.58)}$$

Behavior of dielectrics in static electric field: Polarization of dielectric

Here we briefly describe the behavior of dielectrics or insulators when placed in static electric field. Ideal dielectrics do not contain free charges. As we know, all material media are composed of atoms where a positively charged nucleus (diameter ~ 10^{-15}m) is surrounded by negatively charged electrons (electron cloud has radius ~ 10^{-10}m) moving around the nucleus. Molecules of dielectrics are neutral macroscopically; an externally applied field causes small displacement of the charge particles creating small electric dipoles.These induced dipole moments modify electric fields both inside and outside dielectric material.

Molecules of some dielectric materials posses permanent dipole moments even in the absence of an external applied field. Usually such molecules consist of two or more dissimilar atoms and are called *polar* molecules. A common example of such molecule is water molecule H_2O. In polar molecules the atoms do not arrange themselves to make the net dipole moment zero. However, in the absence of an external field, the molecules arrange themselves in a random manner so that net dipole moment over a volume becomes zero. Under the influence of an applied electric field, these dipoles tend to align themselves along the field as shown in figure 2.15. There are some materials that can exhibit net permanent dipole moment even in the absence of applied field. These materials are called *electrets* that made by heating certain waxes or plastics in the presence of electric field. The applied field aligns the polarized molecules when the material is in the heated state and they are frozen to their new position when after the temperature is brought down to its normal temperatures. Permanent polarization remains without an externally applied field.

As a measure of intensity of polarization, polarization vector $\vec{P}$ (in C/m^2) is defined as:

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_{i=1}^{n\Delta v} \vec{P_k}}{\Delta v} \quad \text{.......................}(2.59)$$

FIGURE MISSING

Fig 2.15: Polarised Dielectric Medium

n being the number of molecules per unit volume i.e. $\vec{P}$ is the dipole moment per unit volume. Let us now consider a dielectric material having polarization $\vec{P}$ and compute the potential at an external point O due to an elementary dipole $\vec{P}\,dv'$.

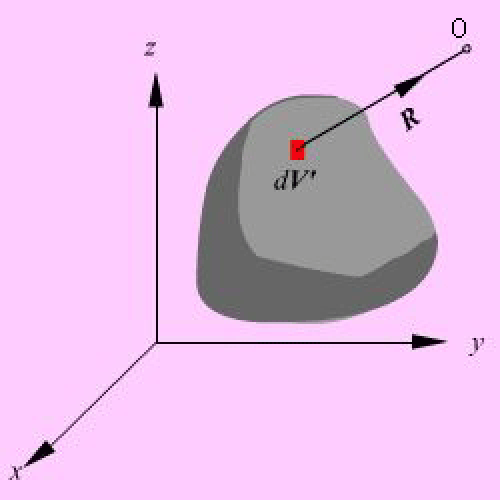


Fig 2.16: Potential at an External Point due to an Elementary Dipole $\vec{P}\,dv'$.

With reference to the figure 2.16, we can write: $dV = \frac{\vec{P}dv' \cdot \hat{a}_R}{4\pi\varepsilon_0 R^2}$ ....*(2.60)*

Therefore,

$V = \int_{v'} \frac{\vec{P}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2} dv'$ ....*(2.61)* $R = \left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{1/2}$ ....*(2.62)*

where x,y,z represent the coordinates of the external point O and x',y',z' are the coordinates of the source point.

From the expression of R, we can verify that

$$\nabla'\left(\frac{1}{R}\right) = \frac{\hat{a}_R}{R^2}$$ ....*(2.63)*

$$V = \frac{1}{4\pi\varepsilon_0}\int_{v'} \vec{P}\cdot\nabla'\left(\frac{1}{R}\right)dv'$$ ....*(2.64)*

Using the vector identity, $\nabla'\cdot(f\vec{A}) = f\nabla'\cdot\vec{A} + \vec{A}\cdot\nabla' f$,where *f* is a scalar quantity , we have,

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{v'}\nabla'\cdot\left(\frac{\vec{P}}{R}\right)dv' - \int_{v'}\frac{\nabla'\cdot\vec{P}}{R}dv'\right]$$ ....*(2.65)*

Converting the first volume integral of the above expression to surface integral, we can write

$$V = \frac{1}{4\pi\varepsilon_0}\oint_{S'} \frac{\vec{P}\cdot\hat{a}'_n}{R} ds' + \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\left(-\nabla\cdot\vec{P}\right)}{R} dv' \quad \text{.................}(2.66)$$

where $\hat{a}'_n$ is the outward normal from the surface element ds' of the dielectric. From the above expression we find that the electric potential of a polarized dielectric may be found from the contribution of volume and surface charge distributions having densities

$$\rho_{ps} = \vec{P}\cdot\hat{a}_n \quad \text{..}(2.67)$$

$$\rho_{pv} = -\nabla\cdot\vec{P} \quad \text{......................}(2.68)$$

These are referred to as polarisation or bound charge densities. Therefore we may replace a polarized dielectric by an equivalent polarization surface charge density and a polarization volume charge density. We recall that bound charges are those charges that are not free to move within the dielectric material, such charges are result of displacement that occurs on a molecular scale during polarization. The total bound charge on the surface is

$$\oint_S \rho_{ps} ds = \oint_S \vec{P}\cdot d\vec{s} \quad \text{......................}(2.69)$$

The charge that remains inside the surface is

$$\int_V \rho_{pv} dv = \int_V -\nabla\cdot\vec{P} dv \quad \text{......................}(2.70)$$

The total charge in the dielectric material is zero as

$$\oint_S \rho_{ps} ds + \int_V \rho_{pv} = \oint_S \vec{P}\cdot d\vec{s} + \int_V -\nabla\cdot\vec{P} dv = \int_V \nabla\cdot\vec{P} - \int_V \nabla\cdot\vec{P} = 0 \quad \text{......................}(2.71)$$

If we now consider that the dielectric region containing charge density ρ_v the total volume charge density becomes

$$\rho_t = \rho_v + \rho_{pv} \quad \text{....................}(2.72)$$

Since we have taken into account the effect of the bound charge density, we can write

$$\nabla\cdot\vec{E} = \frac{\left(\rho_v + \rho_{pv}\right)}{\varepsilon_0} \quad \text{....................}(2.73)$$

Using the definition of ρ_{pv} we have

$$\nabla.(\varepsilon_0 \vec{E} + \vec{P}) = \rho_v \qquad \text{....................}(2.74)$$

Therefore the electric flux density $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

When the dielectric properties of the medium are linear and isotropic, polarisation is directly proportional to the applied field strength and

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \qquad \text{........................}(2.75)$$

is the electric susceptibility of the dielectric. Therefore,

$$\vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \qquad \text{.......................}(2.76)$$

$\varepsilon_r = 1 + \chi_e$ is called relative permeability or the dielectric constant of the medium. $\varepsilon_0 \varepsilon_r$ is called the absolute permittivity.

A dielectric medium is said to be linear when χ_e is independent of $\vec{E}$ and the medium is homogeneous if χ_e is also independent of space coordinates. A linear homogeneous and isotropic medium is called a **simple medium** and for such medium the relative permittivity is a constant.

Dielectric constant ε_r may be a function of space coordinates. For anistropic materials, the dielectric constant is different in different directions of the electric field, D and E are related by a permittivity tensor which may be written as:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \qquad \text{.......................}(2.77)$$

For crystals, the reference coordinates can be chosen along the principal axes, which make off diagonal elements of the permittivity matrix zero. Therefore, we have

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \qquad \text{.......................}(2.78)$$

Media exhibiting such characteristics are called **biaxial**. Further, if $\varepsilon_1 = \varepsilon_2$ then the medium is called **uniaxial**. It may be noted that for isotropic media, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$.

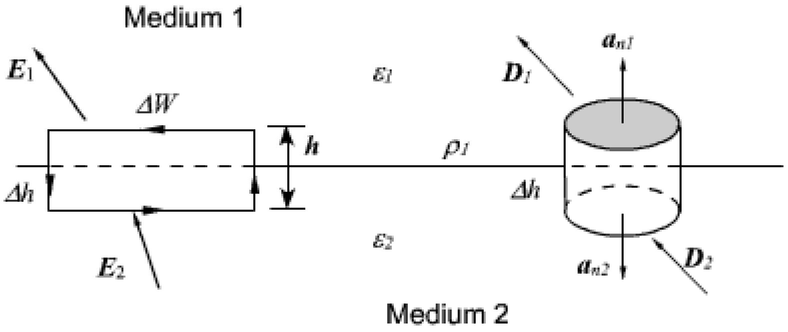
Lossy dielectric materials are represented by a complex dielectric constant, the imaginary part of which provides the power loss in the medium and this is in general dependant on frequency.

Another phenomenon is of importance is **dielectric breakdown**. We observed that the applied electric field causes small displacement of bound charges in a dielectric material that results into polarization. Strong field can pull electrons completely out of the molecules. These electrons being accelerated under influence of electric field will collide with molecular lattice structure causing damage or distortion of material. For very strong fields, avalanche breakdown may also occur. The dielectric under such condition will become conducting.

The maximum electric field intensity a dielectric can withstand without breakdown is referred to as the **dielectric strength** of the material.

Boundary Conditions for Electrostatic Fields:

Let us consider the relationship among the field components that exist at the interface between two dielectrics as shown in the figure 2.17. The permittivity of the medium 1 and medium 2 are ε_1 and ε_2 respectively and the interface may also have a net charge density ρ_s Coulomb/m.



$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$
$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n} \quad ....(2.79)$$

where E_t and E_n are the tangential and normal components of the electric field respectively.

Let us assume that the closed path is very small so that over the elemental path length the variation of E can be neglected. Moreover very near to the interface, $\Delta h \to 0$. Therefore

$$\oint \vec{E}.d\vec{l} = E_{1t}\Delta w - E_{2t}\Delta w + \frac{h}{2}(E_{1n} + E_{2n}) - \frac{h}{2}(E_{1n} + E_{2n}) = 0 \quad ....(2.80)$$

Thus, we have,

$E_{1t} = E_{2t}$ or $\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$ i.e. the **tangential component of an electric field is continuous across the interface**.

For relating the flux density vectors on two sides of the interface we apply Gauss's law to a small pillbox volume as shown in the figure. Once again as $\Delta h \to 0$, we can write

$$\oint \vec{D}\cdot d\vec{s} = (\vec{D_1}.\hat{a}_{n2} + \vec{D_2}\cdot\hat{a}_{n1})\Delta s = \rho_s \Delta s \quad \text{..................(2.81a)}$$

i.e., $D_{1n} - D_{2n} = \rho_s$ ....(2.81b)

.e., $\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$ ....(2.81c)

Thus we find that the **normal component of the flux density vector *D* is discontinuous across an interface by an amount of discontinuity equal to the surface charge density at the interface.**

Example

Two further illustrate these points; let us consider an example, which involves the refraction of D or E at a charge free dielectric interface as shown in the figure 2.18.

Using the relationships we have just derived, we can write

$$E_{1t} = E_1 \sin\theta_1 = \frac{D_1}{\varepsilon_1}\sin\theta_1 = E_{2t} = E_2\sin\theta_2 = \frac{D_2}{\varepsilon_2}\sin\theta_2 \quad \text{.......................(2.82a)}$$

$$D_{1n} = D_1\cos\theta_1 = D_{2n} = D_2 Cos\theta_2 \quad \text{.......................(2.82b)}$$

In terms of flux density vectors,

$$\frac{D_1}{\varepsilon_1}\sin\theta_1 = \frac{D_2}{\varepsilon_2}\sin\theta_2 \quad \text{.......................(2.83a)}$$

$$D_1\cos\theta_1 = D_2 Cos\theta_2 \quad \text{.......................}\mathit{(2.83b)}$$

Therefore, $\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$ ....*(2.84)*

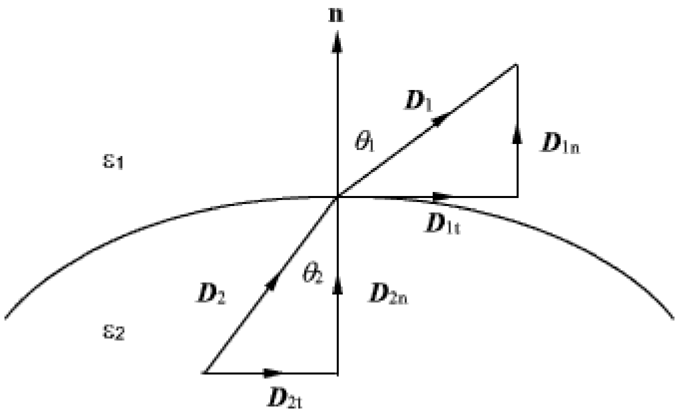


Fig *2*.18: Refraction of D or E at a Charge Free Dielectric Interface

Capacitance and Capacitors

We have already stated that a conductor in an electrostatic field is an Equipotential body and any charge given to such conductor will distribute themselves in such a manner that electric field inside the conductor vanishes. If an additional amount of charge is supplied to an isolated conductor at a given potential, this additional charge will increase the surface charge density ρ_s. Since the potential of the conductor is given by $V = \frac{1}{4\pi\varepsilon_0}\int_{s'} \frac{\rho_s ds'}{r}$, the potential of the conductor will also increase maintaining the ratio $\frac{Q}{V}$ same. Thus we can write $C = \frac{Q}{V}$ where the constant of proportionality C is called the capacitance of the isolated conductor. SI unit of capacitance is Coulomb/ Volt also called Farad denoted by F. It can It can be seen that if V=1, $C = Q$. Thus capacity of an isolated conductor can also be defined as the amount of charge in Coulomb required to raise the potential of the conductor by 1 Volt.

Of considerable interest in practice is a capacitor that consists of two (or more) conductors carrying equal and opposite charges and separated by some dielectric media or free space. The conductors may have arbitrary shapes. A two-conductor capacitor is shown in figure 2.19.

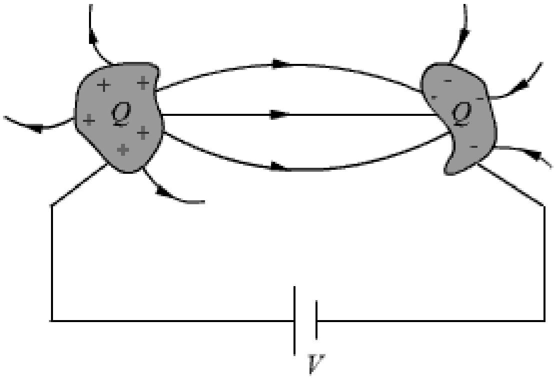


Fig 2.19: Capacitance and Capacitors

When a d-c voltage source is connected between the conductors, a charge transfer occurs which results into a positive charge on one conductor and negative charge on the other conductor. The conductors are equipotential surfaces and the field lines are perpendicular to the conductor surface. If V is the mean potential difference between the conductors, the capacitance is given by $C = \frac{Q}{V}$. Capacitance of a capacitor depends on the geometry of the conductor and the permittivity of the medium between them and does not depend on the charge or potential difference between conductors. The capacitance can be computed by assuming Q(at the same time -Q on the other conductor), first determining $\vec{E}$ using Gauss's theorem and then determining $V = -\int \vec{E}.d\vec{l}$. We illustrate this procedure by taking the example of a parallel plate capacitor.

Example: Parallel plate capacitor

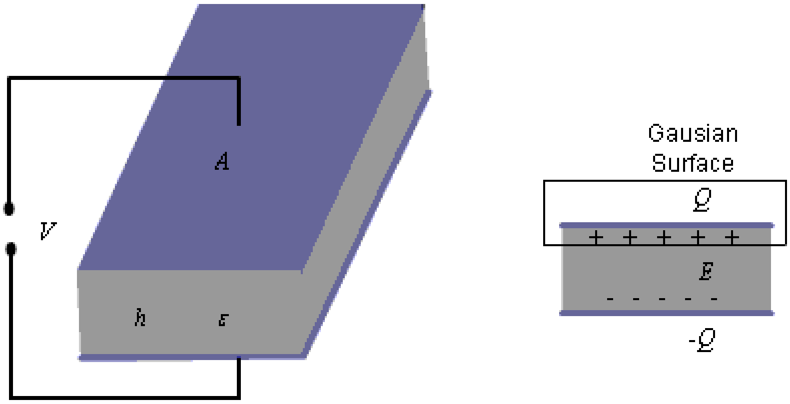


Fig 2.20: Parallel Plate Capacitor

For the parallel plate capacitor shown in the figure 2.20, let each plate has area A and a distance h separates the plates. A dielectric of permittivity ε fills the region between the plates. The electric field lines are confined between the plates. We ignore the flux fringing at the edges of the plates and charges are assumed to be uniformly distributed over the conducting plates with densities ρ_s and -ρ_s, $\rho_s = \frac{Q}{A}$.

By Gauss's theorem we can write, $E = \frac{\rho_s}{\varepsilon} = \frac{Q}{A\varepsilon}$ ....*(2.85)*

As we have assumed ρ_s to be uniform and fringing of field is neglected, we see that E is constant in the region between the plates and therefore, we can write $V = Eh = \frac{hQ}{\varepsilon A}$. Thus, for a parallel plate capacitor we have, $C = \frac{Q}{V} = \varepsilon \frac{A}{h}$ ....*(2.86)*

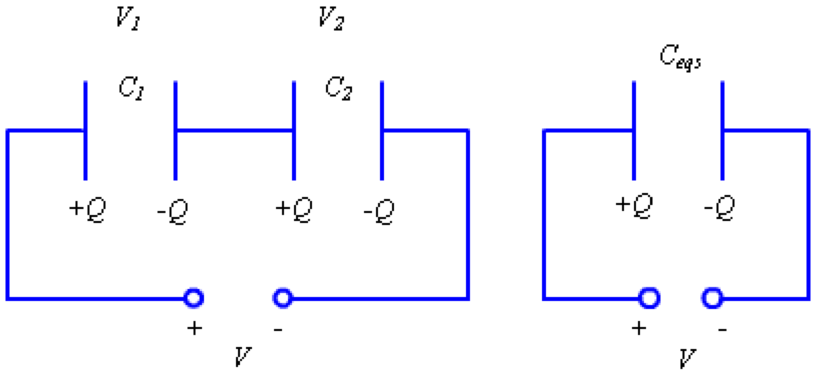
Series and parallel Connection of capacitors

Capacitors are connected in various manners in electrical circuits; series and parallel connections are the two basic ways of connecting capacitors. We compute the equivalent capacitance for such connections.

Series Case: Series connection of two capacitors is shown in the figure 2.21. For this case we can write,

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{V}{Q} = \frac{1}{C_{eqs}} = \frac{1}{C_1} + \frac{1}{C_2} \qquad \text{.......................(2.87)}$$



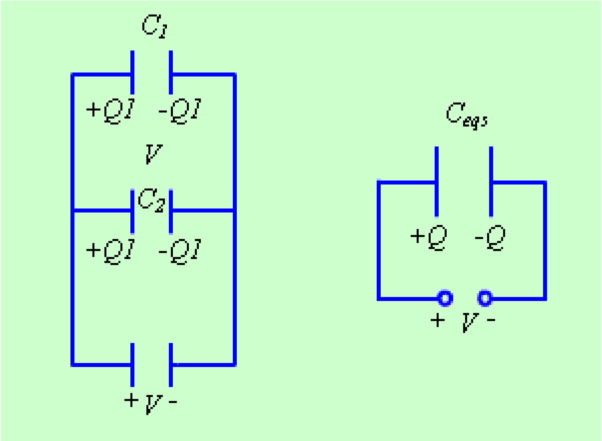


Fig 2.22: Parallel Connection of Capacitors

The same approach may be extended to more than two capacitors connected in series.

Parallel Case: For the parallel case, the voltages across the capacitors are the same.

The total charge $Q = Q_1 + Q_2 = C_1V + C_2V$

Therefore, $$C_{eq} = \frac{Q}{V} = C_1 + C_2 \qquad \text{.......................(2.88)}$$

Electrostatic Energy and Energy Density

We have stated that the electric potential at a point in an electric field is the amount of work required to bring a unit positive charge from infinity (reference of zero potential) to that point. To determine the energy that is present in an assembly of charges, let us first determine the amount of work required to assemble them. Let us consider a number of discrete charges Q_1, Q_2,......., Q_N are brought from infinity to their present position one by one. Since initially there is no field present, the amount of work done in bring Q_1 is zero. Q_2 is brought in the presence of the field of Q_1, the work done $W_1 = Q_2V_{21}$ where V_{21} is the potential at the location of Q_2 due to Q_1. Proceeding in this manner, we can write, the total work done

$$W = V_{21}Q_2 + (V_{31}Q_3 + V_{32}Q_3) + \ldots\ldots\ldots\ldots + (V_{N1}Q_N + \ldots\ldots + V_{N(N-1)}Q_N)$$
....(2.89)

Had the charges been brought in the reverse order,

$$W = (V_{1N}Q_1 + \ldots\ldots + V_{12}Q_1) + \ldots\ldots\ldots\ldots\ldots\ldots + (V_{(N-2)(N-1)}Q_{N-2} + V_{(N-2)N}Q_{N-2}) + V_{(N-1)N}Q_{N-1}$$
.................(2.90)

Therefore,

$$2W = (V_{1N} + V_{1(N-1)} + \ldots\ldots + V_{12})Q_1 + (V_{2N} + V_{2(N-1)} + \ldots\ldots + V_{23} + V_{21})Q_1 \ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots + (V_{N1} + \ldots\ldots V_{N2} + V_{N(N-1)})Q_N$$
................*(2.91)*

Here V_{IJ} represent voltage at the I^{th} charge location due to J^{th} charge. Therefore,

$$2W = V_1Q_1 + \ldots\ldots\ldots\ldots + V_NQ_N = \sum_{I=1}^{N} V_IQ_I$$

Or, $$W = \frac{1}{2}\sum_{I=1}^{N} V_IQ_I \qquad \text{................(2.92)}$$

If instead of discrete charges, we now have a distribution of charges over a volume v then we can write,

$$W = \frac{1}{2}\int_v V\rho_v dv \qquad \text{................(2.93)}$$

where ρ_v is the volume charge density and V represents the potential function.

Since, $\rho_v = \nabla \cdot \vec{D}$, we can write

$$W = \frac{1}{2}\int_v \left(\nabla \cdot \vec{D}\right) V dv \quad \text{.......................................}(2.94)$$

Using the vector identity,

$\nabla.(V\vec{D}) = \vec{D}\cdot\nabla V + V\nabla\cdot\vec{D}$, we can write

$$\begin{aligned} W &= \frac{1}{2}\int_v \left(\nabla.(V\vec{D}) - \vec{D}\cdot\nabla V\right) dv \\ &= \frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s} - \frac{1}{2}\int_v \left(\vec{D}\cdot\nabla V\right) dv \end{aligned} \quad \text{................}(2.95)$$

In the expression $\frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s}$, for point charges, since V varies as $\frac{1}{r}$ and D varies as $\frac{1}{r^2}$, the term $V\vec{D}$ varies as $\frac{1}{r^3}$ while the area varies as r^2. Hence the integral term varies at least as $\frac{1}{r}$ and the as surface becomes large (i.e. $r \to \infty$) the integral term tends to zero.

Thus the equation for W reduces to

$$W = -\frac{1}{2}\int_v \left(\vec{D}\nabla V\right) dv = \frac{1}{2}\int_v \left(\vec{D}.\vec{E}\right) dv = \frac{1}{2}\int_v \left(\varepsilon E^2\right) dv = \int_v w_e dv \quad \text{................}(2.96)$$

$w_e = \frac{1}{2}\varepsilon E^2$, is called the energy density in the electrostatic field.

Poisson's and Laplace's Equations

For electrostatic field, we have seen that

$$\begin{aligned} &\nabla\cdot\vec{D} = \rho_v \\ &\vec{E} = -\nabla V \end{aligned} \quad \text{..}(2.97)$$

Form the above two equations we can write

$\nabla\cdot(\varepsilon\vec{E}) = \nabla\cdot(-\varepsilon\nabla V) = \rho_v$ ....(2.98)

Using vector identity we can write, $\varepsilon\nabla\cdot\nabla V + \nabla V\cdot\nabla\varepsilon = -\rho_v$ ....(2.99)

For a simple homogeneous medium, ε is constant and $\nabla\varepsilon = 0$. Therefore,

$$\nabla\cdot\nabla V = \nabla^2 V = -\frac{\rho_v}{\varepsilon}$$ ....(2.100)

This equation is known as **Poisson's equation**. Here we have introduced a new operator, ∇^2 (del square), called the Laplacian operator. In Cartesian coordinates,

$$\nabla^2 V = \nabla\cdot\nabla V = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\cdot\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = -\frac{\rho_v}{\varepsilon}$$ ....(2.102)

In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$\nabla^2 V = 0$$ ....*(2.105)*

which is known as Laplace's equation.

Laplace's and Poisson's equation are very useful for solving many practical electrostatic field problems where only the electrostatic conditions (potential and charge) at some boundaries are known and solution of electric field and potential is to be found throughout the volume. We shall consider such applications in the section where we deal with boundary value problems.

ASSIGNMENT PROBLEMS

1. A charged ring of radius d carrying a charge of ρ_l C/m lies in the x-y plane with its centre at the origin and a charge Q C is placed at the point $(0,0,2d)$. Determine ρ_l in terms of Q and d so that a test charge placed at $(0,0,2d)$ does not experience any force.

2. A semicircular ring of radius a lies in the free space and carries a charge density ρ_l C/m. Find the electric field at the centre of the semicircle.
3. Consider a uniform sphere of charge with charge density ρ_0 and radius b, centered at the origin. Find the electric field at a distance r from the origin for the two cases: $r<b$ and $r>b$. Sketch the strength of the electric filed as function of r.

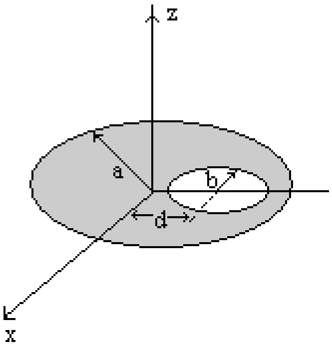
4. A spherical charge distribution is given by

$$\rho_v = \begin{cases} \rho_0 (a^2 - r^2), & r \le a \\ 0, & r > a \end{cases}$$

a is the radius of the sphere. Find the following:

i. The total charge.
ii. $\vec{E}$ for $r \le a$ and $r > a$.
iii. The value of r where the $\vec{E}$ becomes maximum.

5. With reference to the Figure 2.6 determine the potential and field at the point $P(0,0,h)$ if the shaded region contains uniform charge density ρ_s/m^2.



L, radius of the inner conductor a and that of outer conductor b. A dielectric material having dielectric