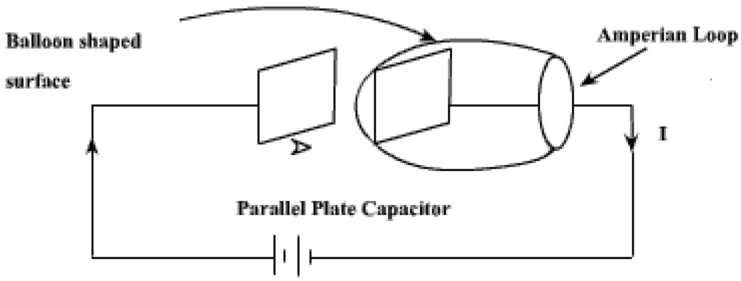
A classic example for this is given below .

Suppose we are in the process of charging up a capacitor as shown in fig 5.3.



$$\nabla.\left(\nabla\times\vec{H}\right)=0=\nabla.\vec{J}+\frac{\partial\rho}{\partial t}$$

$$=\nabla.\vec{J}+\frac{\partial}{\partial t}\nabla.\vec{D}$$

$$=\nabla.\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right) \tag{5.23}$$

$$\therefore\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \tag{5.24}$$

The equation (5.24) is valid for static as well as for time varying case.

Equation (5.24) indicates that a time varying electric field will give rise to a magnetic field even in the absence of $\vec{J}$. The term $\frac{\partial\vec{D}}{\partial t}$ has a dimension of current densities $\left(A/m^2\right)$ and is called the displacement current density.

Introduction of $\frac{\partial\vec{D}}{\partial t}$ in $\nabla\times\vec{H}$ equation is one of the major contributions of Jame's Clerk Maxwell. The modified set of equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (5.25a)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (5.25b)$$

$$\nabla . \vec{D} = \rho \qquad (5.25c)$$

$$\nabla . \vec{B} = 0 \qquad (5.25d)$$

is known as the Maxwell's equation and this set of equations apply in the time varying scenario, static fields are being a particular case $\left(\frac{\partial}{\partial t} = 0 \right)$.

In the integral form

$$\oint_c \vec{E}.d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t}.d\vec{S} \qquad (5.26a)$$

$$\oint_c \vec{H}.d\vec{l} = \int_s \left(J + \frac{\partial D}{\partial t} \right).d\vec{S} = I + \int_s \frac{\partial \vec{D}}{\partial t}.d\vec{S} \qquad (5.26b)$$

$$\int_V \nabla.\vec{D} dv = \oint_S \vec{D}.d\vec{S} = \int_V \rho dv \qquad (5.26c)$$

$$\oint \vec{B}.d\vec{S} = 0 \qquad (5.26d)$$

The modification of Ampere's law by Maxwell has led to the development of a unified electromagnetic field theory. By introducing the displacement current term, Maxwell could predict the propagation of EM waves. Existence of EM waves was later demonstrated by Hertz experimentally which led to the new era of radio communication.

Boundary Conditions for Electromagnetic fields

The differential forms of Maxwell's equations are used to solve for the field vectors provided the field quantities are single valued, bounded and continuous. At the media boundaries, the field vectors are discontinuous and their behaviors across the boundaries are governed by boundary conditions. The integral equations(eqn 5.26) are assumed to hold for regions containing discontinuous media.Boundary conditions can be derived by applying the Maxwell's equations in the integral form to small regions at the interface of the two media. The procedure is similar to those used for obtaining boundary conditions for static electric fields (chapter 2) and static magnetic fields (chapter 4). The boundary conditions are summarized as follows

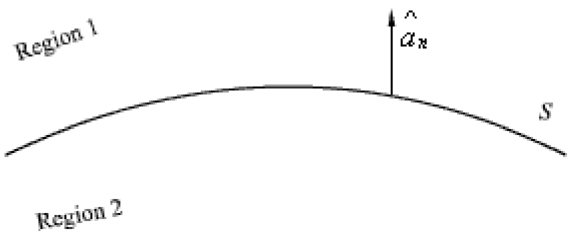
With reference to fig 5.3

$$\hat{a}_n \times \left(\vec{E}_1 - \vec{E}_2\right) = 0 \qquad 5.27(a)$$

$$\hat{a}_n \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_S \qquad 5.27(b)$$

$$\hat{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_S \qquad 5.27(c)$$

$$\hat{a}_n \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \qquad 5.27(d)$$



$\vec{D}$ is discontinuous across the interface by an amount equal to the surface current density while normal component of the magnetic flux density is continuous.
If one side of the interface, as shown in fig 5.4, is a perfect electric conductor, say region 2, a surface current $\vec{J}_S$ can exist even though $\vec{E}$ is zero as $\sigma = \infty$.
Thus eqn 5.27(a) and (c) reduces to

$$\hat{a}_n \times \vec{H} = \vec{J}_S \qquad (5.28(a))$$

$$\hat{a}_n \times \vec{E} = 0 \qquad (5.28(b))$$

Wave equation and their solution:

From equation 5.25 we can write the Maxwell's equations in the differential form as

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = \vec{\rho}$$

$$\nabla \cdot \vec{B} = 0$$

Let us consider a source free uniform medium having dielectric constant ε, magnetic permeability μ and conductivity σ. The above set of equations can be written as

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad (5.29(a))$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad (5.29(b))$$

$$\nabla \cdot \vec{E} = 0 \qquad (5.29(c))$$

$$\nabla \cdot \vec{H} = 0 \qquad (5.29(d))$$

Using the vector identity ,

$$\nabla \times \nabla \times \vec{A} = \nabla \cdot \left(\nabla \cdot \vec{A}\right) - \nabla^2 A$$

We can write from 5.29(b)

$$\nabla \times \nabla \times \vec{E} = \nabla \cdot \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

$$= -\nabla \times \left(\mu \frac{\partial \vec{H}}{\partial t} \right)$$

or $$\nabla \cdot \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

Substituting $\nabla \times \vec{H}$ from 5.29(a)

$$\nabla \cdot \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right)$$

But in source free medium $\nabla \cdot \vec{E} = 0$ (eqn 5.29(c))

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad (5.30)$$

In the same manner for equation eqn 5.29(a)

$$\nabla \times \nabla \times \vec{H} = \nabla \cdot \left(\nabla \cdot \vec{H}\right) - \nabla^2 \vec{H}$$

$$= \sigma\left(\nabla \times \vec{E}\right) + \varepsilon \frac{\partial}{\partial t}\left(\nabla \times \vec{E}\right)$$

$$= \sigma\left(-\mu \frac{\partial \vec{H}}{\partial t}\right) + \varepsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \vec{H}}{\partial t}\right)$$

Since $\nabla \cdot \vec{H} = 0$ from eqn 5.29(d), we can write

$$\nabla^2 \vec{H} = \mu\sigma\left(\frac{\partial \vec{H}}{\partial t}\right) + \mu\varepsilon\left(\frac{\partial^2 \vec{H}}{\partial t^2}\right) \qquad (5.31)$$

These two equations

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu\sigma\left(\frac{\partial \vec{H}}{\partial t}\right) + \mu\varepsilon\left(\frac{\partial^2 \vec{H}}{\partial t^2}\right)$$

are known as wave equations.

It may be noted that the field components are functions of both space and time. For example, if we consider a Cartesian co ordinate system, $\vec{E} \text{ and } \vec{H}$ essentially represents $\vec{E}(x,y,z,t)$ and $\vec{H}(x,y,z,t)$. For simplicity, we consider propagation in free space , i.e. $\sigma = 0$, $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$. The wave eqn in equations 5.30 and 5.31 reduces to

$$\nabla^2 \vec{E} = \mu_0\varepsilon_0\left(\frac{\partial^2 \vec{E}}{\partial t^2}\right) \qquad (5.32(a))$$

$$\nabla^2 \vec{H} = \mu_0\varepsilon_0\left(\frac{\partial^2 \vec{H}}{\partial t^2}\right) \qquad (5.32(b))$$

Further simplifications can be made if we consider in Cartesian co ordinate system a special case where $\vec{E} \text{ and } \vec{H}$ are considered to be independent in two dimensions, say $\vec{E} \text{ and } \vec{H}$ are assumed to be independent of y and z. Such waves are called plane waves.

From eqn (5.32 (a)) we can write

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

The vector wave equation is equivalent to the three scalar equations

$$\frac{\partial^2 \vec{E_x}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_x}}{\partial t^2} \right) \qquad (5.33(a))$$

$$\frac{\partial^2 \vec{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_y}}{\partial t^2} \right) \qquad (5.33(b))$$

$$\frac{\partial^2 \vec{E_z}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_z}}{\partial t^2} \right) \qquad (5.33(c))$$

Since we have $\nabla \cdot \vec{E} = 0$,

$$\therefore \frac{\partial \vec{E_x}}{\partial x} + \frac{\partial \vec{E_y}}{\partial y} + \frac{\partial \vec{E_z}}{\partial z} = 0 \qquad (5.34)$$

As we have assumed that the field components are independent of y and z eqn (5.34) reduces to

$$\frac{\partial E_x}{\partial x} = 0 \qquad (5.35)$$

i.e. there is no variation of E_x in the x direction.

Further, from 5.33(a), we find that $\frac{\partial E_x}{\partial x} = 0$ implies $\frac{\partial^2 E_x}{\partial t^2} = 0$ which requires any three of the conditions to be satisfied: (i) $E_x=0$, (ii)E_x = constant, (iii)E_x increasing uniformly with time.

A field component satisfying either of the last two conditions (i.e (ii) and (iii))is not a part of a plane wave motion and hence E_x is taken to be equal to zero. Therefore, a uniform plane wave propagating in x direction does not have a field component (E or H) acting along x.

Without loss of generality let us now consider a plane wave having E_y component only (Identical results can be obtained for E_z component) .

The equation involving such wave propagation is given by

$$\frac{\partial^2 \overrightarrow{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \overrightarrow{E_y}}{\partial t^2} \right) \qquad (5.36)$$

The above equation has a solution of the form

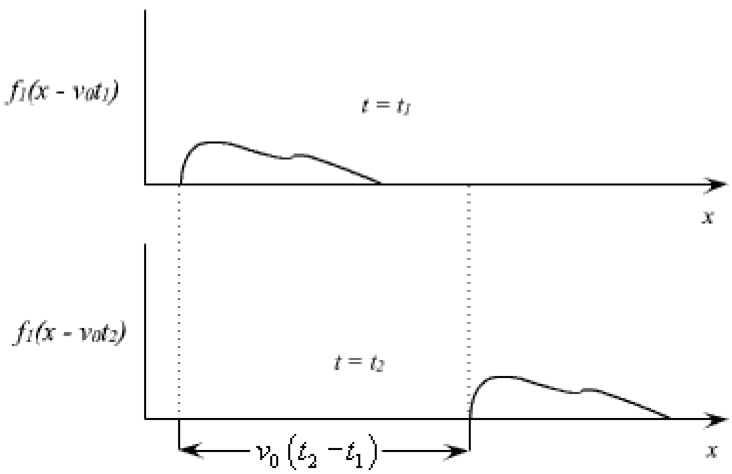
$$E_y = f_1(x - v_0 t) + f_2(x + v_0 t) \qquad (5.37)$$

where $v_0 = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Thus equation (5.37) satisfies wave eqn (5.36) can be verified by substitution. $f_1(x - v_0 t)$ corresponds to the wave traveling in the + x direction while $f_2(x + v_0 t)$ corresponds to a wave traveling in the -x direction. The general solution of the wave eqn thus consists of two waves, one traveling away from the source and other traveling back towards the source. In the absence of any reflection, the second form of the eqn (5.37) is zero and the solution can be written as

$$E_y = f_1(x - v_0 t) \qquad (5.38)$$

Such a wave motion is graphically shown in fig 5.5 at two instances of time t_1 and t_2.



$\vec{E} = \widehat{a_y} E_y = \widehat{a_y} f_1(x - v_0 t)$ and there is no variation along y and z.

$$\nabla \times \vec{E} = \widehat{a_z} \frac{\partial E_y}{\partial x}$$

Since only z component of $\nabla \times \vec{E}$ exists, from (5.29(b))

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \qquad (5.39)$$

and from (5.29(a)) with $\sigma = 0$, only H_z component of magnetic field being present

$$\nabla \times \vec{H} = -\hat{a}_y \frac{\partial H_z}{\partial x}$$

$$\therefore -\frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{\partial E_y}{\partial t} \qquad (5.40)$$

Substituting E_y from (5.38)

$$\frac{\partial H_z}{\partial x} = -\varepsilon_0 \frac{\partial E_y}{\partial t} = \varepsilon_0 v_0 f_1'(x - v_0 t)$$

$$\therefore \frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}} f_1'(x - v_0 t)$$

$$\therefore H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \int f_1'(x - v_0 t) dx + c$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} \int \frac{\partial}{\partial x} f_1 dx + c$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} f_1 + c$$

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y + c$$

The constant of integration means that a field independent of x may also exist. However, this field will not be a part of the wave motion.

Hence
$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y \qquad (5.41)$$

which relates the E and H components of the traveling wave.

$$z_0 = \frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi \text{ or } 377\Omega$$

$z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is called the characteristic or intrinsic impedance of the free space

ASSIGNMENT PROBLEMS

1. A rectangular loop of area $a \times b \;\; \mathrm{m}^2$ rotates at ω rad/s in a magnetic fields of B Wb/m^2 normal to the axis of rotation. If the loop has N turns determine the induced voltage in the loop.

2. If the electric field component in a nonmagnetic dielectric medium is given by

$$\vec{E} = 50\log\left(10^9 t - 8x\right)\hat{a}_y$$

determine the dielectric constant and the corresponding $\vec{H}$.

3. A vector field $\vec{A}$ in phasor form is given by

$$\vec{A} = j5ye^{-jy}\hat{a}_x$$

Express $\vec{A}$ in instantaneous form.

Unit V Electromagnetic waves

In the previous chapter we introduced the equations pertaining to wave propagation and discussed how the wave equations are modified for time harmonic case. In this chapter we discuss in detail a particular form of electromagnetic wave propagation called 'plane waves'.

The Helmhotz Equation:

In source free linear isotropic medium, Maxwell equations in phasor form are,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{E} = 0$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \qquad \nabla \times \vec{H} = 0$$

$$\therefore \nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \times \vec{E}\right) - \nabla^2\vec{E} = -j\omega\mu\nabla \times \vec{H}$$

or, $$-\nabla^2\vec{E} = -j\omega\mu(j\omega\varepsilon\vec{E})$$

or, $$\nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = 0$$