Z TRANSFORM

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Z TRANSFORM

The bilateral (or) two sided Z-transform of a discrete time signal (or) sequence $x(n)$ is defined as,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

where Z is a complex variable

The one sided (or) unilateral Z-transform is defined as

$$X(z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

Relation between DTFT and Z-transform

The DTFT of a sequence $x(n)$ is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

For the existence of DTFT, the above summation should converge i.e., $x(n)$ must be absolutely summable.

The Z-transform of the sequence $x(n)$ is given by

$$Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

Where Z is a complex variable

Z is given by

$$Z = r e^{j\omega}$$

Where r is the radius of a circle

$$\therefore X(Z) = X(r e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) (r e^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n}$$

For the existence of Z-transform, the above summation should converge i.e., $x(n)r^{-n}$ must be absolutely summable.

$$\text{i.e., } \sum_{n=-\infty}^{\infty} [x(n)r^{-n}] < \alpha$$

The above equation represents the DTFT of a signal $x(n)r^{-n}$. Hence we can say that the Z-transform of $x(n)$ is same as the DTFT of $x(n)r^{-n}$.

For the DTFT to exist, the discrete sequence x(n) must be absolutely summable.

Region of convergence (ROC)

For any given sequence, the Z-transform (or) may not converge. The set of values of Z (or) the set of points in Z-plane for which X(Z) converges is called the Region of convergence (ROC) of X(Z).

Properties of ROC

1. The ROC of X(Z) is a ring (or) disk in Z-plane, with centre at the origin.
2. The ROC cannot contain any poles.
3. If $x(n)$ is finite duration right sided (casual) sequence then the ROC is entire Z-plane except $Z = 0$.
4. If $x(n)$ is finite duration left sided (non casual) sequence then the ROC is entire Z-plane except $Z = \infty$.
5. If $x(n)$ is finite duration two sided sequence then the ROC is entire Z-plane except $Z = 0$ and $Z = \infty$.

6. If $x(n)$ is infinite duration right sided (causal) sequence then the ROC is exterior of a circle of radius r_1.

7. If $x(n)$ is infinite duration left sided (non causal) sequence then the ROC is interior of a circle of radius r_2.

8. If $x(n)$ is infinite duration two sided sequence, then the ROC is the region in between two circles of radius r_1 and r_2.

9. If the Z-transform is rational, then the ROC will extend to infinity. This means that the ROC will be bounded by poles.

10. The ROC of an LTI stable system contains the unit circle.

1. Find the ROC and Z-transform of following sequences.

(a) $x(n) = \left\{\underset{\uparrow}{1}, 0, 3, 5, 4\right\}$ **(b)** $x(n) = \{5, 3, -2, 4\}$

(c) $x(n) = \left\{3, 2, -1, 1, \underset{\uparrow}{-2}\right\}$ **(d)** $x(n) = \left\{2, 1, -4, \underset{\uparrow}{2}, 1, 5\right\}$

Solution

(a) $x(n) = \left\{\underset{\uparrow}{1}, 0, 3, 5, 4\right\}$

The given sequence values are,

$x(0) = 1, x(1) = 0, x(2) = 3, x(3) = 5, x(4) = 4$

$$\text{w.k.t, } X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$= x(0) + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3} + x(4) Z^{-4}$$

$$X(Z) = 1 + 3Z^{-2} + 5Z^{-3} + 4Z^{-4}$$

ROC: Entire Z-plane except at $Z = 0$.

(b) $x(n) = \{5, 3, -2, 4\}$

The given sequence values are

$$x(0) = 5,\ x(1) = 3,\ x(2) = -2,\ x(3) = 4$$

$$\text{w.k.t, } X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$= x(0) + x(1)\, Z^{-1} + x(2)\, Z^{-2} + x(3)\, Z^{-3}$$

$$X(Z) = 5 + 3Z^{-1} - 2Z^{-2} + 4Z^{-3}$$

ROC: Entire Z-plane except at $Z = 0$.

(c) $x(n) = \left\{3,\ 2,\ -1,\ 1,\ \underset{\uparrow}{-2}\right\}$

The given sequence values are,

$x(-4) = 3,\ x(-3) = 2,\ x(-2) = -1,\ x(-1) = 1,\ x(0) = -2$

$$\text{w.k.t, } X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$= x(-4)Z^4 + x(-3)Z^3 + x(-2)Z^2 + x(-1)Z^1 + x(0)$$

$$= 3Z^4 + 2Z^3 - 1Z^2 + 1Z^1 - 2$$

$$X(Z) = 3Z^4 + 2Z^3 - Z^2 + Z - 2$$

ROC: Entire Z-plane except at $Z = \infty$.

(d) $x(n) = \left\{2, 1, -4, \underset{\uparrow}{2}, 1, 5\right\}$

The given sequence values are,

$x(-3) = 2, x(-2) = 1, x(-1) = -4, x(0) = 2, x(1) = 1, x(2) = 5$

w.k.t, $X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$

$= x(-3)Z^3 + x(-2)Z^2 + x(-1)Z^1 + x(0) + x(1)Z^{-1} + x(2)Z^{-2}$

$= 2Z^3 + 1Z^2 - 4Z + 2 + 1Z^{-1} + 5Z^{-2}$

$X(Z) = 2Z^3 + Z^2 - 4Z + 2 + Z^{-1} + 5Z^{-2}$

ROC: Entire Z-plane except at $Z = 0$ and $Z = \infty$.

1. Linearity property

The linearity property of Z-transform states that Z-transform of a weighted sum of two signals is equal to the weighted sum of individual Z-transforms.

It states that

If $Z[x_1(n)] = X_1(Z)$ and $Z[x_2(n)] = X_2(Z)$

then $Z[ax_1(n) + bx_2(n)] = aX_1(Z) + bX_2(Z)$

Proof:

$$Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n)\, Z^{-n}$$

$$Z[a\,x_1(n) + b\,x_2(n)] = \sum_{n=-\infty}^{\infty} [a\,x_1(n) + b\,x_2(n)]Z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n)Z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n)Z^{-n}$$

$$= a\,X_1(Z) + b\,X_2(Z).$$

2. Time shifting property

The time shifting property of Z-transform states that,

If $Z[x(n)] = X(Z)$ with zero initial conditions

then $Z[x(n-m)] = Z^{-m} X(Z)$

Proof:

$$Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) Z^{-n}$$

Let $n - m = P$, then $n = m + P$

$$Z[x(n-m)] = \sum_{P=-\infty} x(P) Z^{-(m+P)}$$

$$= Z^{-m} \sum_{P=-\infty}^{\infty} x(P) Z^{-P}$$

$$Z[x(n-m)] = Z^{-m} X(Z)$$

Similarly,

$Z[x(n+m)] = Z^{m} X(Z)$.

If the initial conditions are not neglected, we have

3. Scaling property (or) Multiplication by an exponential sequence property

It states that

If $Z[x(n)] = X(Z)$

then $Z[a^n x(n)] = X\left(\frac{Z}{a}\right) = X(a^{-1}Z)$

Proof:

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left(\frac{Z}{a}\right)^{-n}$$

$$= X\left(\frac{Z}{a}\right) = X(a^{-1}Z)$$

Note:

$$Z\left[e^{j\omega n}\, x(n)\right] = X\left(\frac{Z}{e^{j\omega}}\right) = X\left(e^{-j\omega}Z\right)$$

$$Z\left[e^{-j\omega n}\, x(n)\right] = X\left(\frac{Z}{e^{-j\omega}}\right) = X\left(e^{j\omega}Z\right)$$

4. Time Reversal property

It states that

If $Z[x(n)] = X(Z)$

then $Z[x(-n)] = X\left(\frac{1}{Z}\right)$

Proof:

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, Z^{-n}$$

Let $P = -n$ in the summation,

$$Z[x(-n)] = \sum_{P=-\infty}^{\infty} x(P)\, Z^{P}$$

$$= \sum_{P=-\infty}^{\infty} x(P)\left(Z^{-1}\right)^{P}$$

$$= X\left(Z^{-1}\right) = X\left(\frac{1}{Z}\right)$$

5. Time Expansion property

It states that

If $Z[x(n)] = X(Z)$

then $Z[x_k(n)] = X(Z^k)$

where $x_k(n) = \begin{cases} x\left(\frac{n}{k}\right), & \text{if n is an integer multiple of k} \\ 0, & \text{otherwise} \end{cases}$

Proof:

$$Z[x_k(n)] = \sum_{n=-\infty}^{\infty} x_k(n) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{k}\right) Z^{-n}$$

Let $\frac{n}{k} = P$ in the above summation

$$Z[x_k(n)] = \sum_{P=-\infty}^{\infty} x(P) Z^{-Pk}$$

$$= \sum_{P=-\infty}^{\infty} x(P)\left(Z^k\right)^{-P} = X(Z^k)$$

6. Multiplication by n (or) Differentiation in Z-domain property

It states that

If $Z[x(n)] = X(Z)$

then $Z[n\,x(n)] = -Z\frac{d}{dZ}X(Z)$

Proof:

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

Differentiating both sides with respect to Z.

$$\frac{d}{dZ}[X(Z)] = \frac{d}{dZ}\left[\sum_{n=-\infty}^{\infty} x(n)Z^{-n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dZ}[Z^{-n}]$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-n)Z^{-n-1}$$

$$= -Z^{-1} \sum_{n=-\infty}^{\infty} [n\,x(n)]Z^{-n}$$

$$\frac{d}{dZ}[X(Z)] = -Z^{-1}\,Z[n\,x(n)] \qquad \text{Note: } Z[n^k\,x(n)] = (-1)^k\,Z^k\,\frac{d^k}{dZ^k}X(Z)$$

$$\therefore \quad Z[n\,x(n)] = -Z\frac{d}{dZ}[X(Z)]$$

Convolution property (or) theorem

It states that Z-transform of the convolution of two signals is equal to the multiplication of their Z-transform.

If $Z[x_1(n)] = X_1(Z)$ and $Z[x_2(n)] = X_2(Z)$

then $Z[x_1(n) * x_2(n)] = X_1(Z)\, X_2(Z)$

Proof:

$$\text{w.k.t, } x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Let $x(n) = x_1(n) * x_2$

$$Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{K=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} x_1(k)\, x_2(n-k)] Z^{-(n-k)}\, Z^{-k}$$

Interchanging the order of summations,

$$X(Z) = \sum_{K=-\infty}^{\infty} x_1(k) Z^{-k} \sum_{n=-\infty}^{\infty} x_2(n-k) Z^{-(n-k)}$$

Let $n - K = p$ in the second summation,

$$X(Z) = \sum_{K=-\infty}^{\infty} x_1(k) Z^{-k} \sum_{p=-\infty}^{\infty} x_2(p) Z^{-p}$$

$$= X_1(Z)\, X_2(Z)$$

1. Find the Z-transform of the sequence $x(n) = a^{n-2}\, u(n-2)$.

Solution:

Given $x(n) = a^{n-2}\, u(n-2)$

The Z-transform of the sequence $x(n) = a^n\, u(n)$ is given by

$$X(Z) = \frac{Z}{Z-a};\quad \text{ROC:}|Z| > |a|$$

Using the time shifting property,

$$Z[x(n-m)] = Z^{-m}\, X(Z)$$

$$\therefore\ Z\left[a^{n-2}\, u(n-2)\right] = Z^{-2}\left[\frac{Z}{Z-a}\right]$$

$$X(Z) = \frac{1}{Z(Z-a)};\quad \text{ROC:}|Z| > |a|$$

2. Find the Z-transform of the signal $x(n) = 3(4)^n u(-n)$

Solution:

Given $x(n) = 3(4)^n u(-n)$

w.k.t, $Z[u(n)] = \dfrac{1}{1 - Z^{-1}}$; ROC:$|Z| > 1$

Using the time reversal property,

$$Z[u(-n)] = Z[u(n)]\big|_{Z \to \frac{1}{2}} = \frac{1}{1 - Z}; \quad \text{ROC:}|Z| > 1$$

Using scaling property

$$Z[(4)^n u(-n)] = Z[u(-n)]\big|_{Z \to \frac{Z}{4}} = \frac{1}{1 - Z}\bigg|_{Z \to \frac{Z}{4}}$$

Using the linearity property,

$$= \frac{1}{1 - \frac{Z}{4}}; \quad \text{ROC:}|Z| < 4$$

$$Z[3(4)^n u(-n)] = 3Z[4^n u(-n)] = \frac{3}{1 - \frac{Z}{4}}; \quad \text{ROC:}|Z| < 4$$

3. Find the Z-transform of

(a) $x(n) = \delta(n+1)$ **(b)** $x(n) = 3\delta(n+2) + 2\delta(n) + \delta(n-1) - \delta(n-2)$

Solution:

(a) Given $x(n) = \delta(n+1)$

w.k.t, $Z[\delta(n)] = 1$

Using the time shifting property,

$$Z[x(n+K)] = Z^K X(Z)$$

$$Z[\delta(n+1)] = Z \cdot 1 = Z$$

(b) Given $x(n) = 3\delta(n+2) + 2\delta(n) + \delta(n-1) - \delta(n-2)$

$$Z[x(n)] = Z[3\delta(n+2) + 2\delta(n) + \delta(n-1) - \delta(n-2)]$$

Using the linearity property,

$$X(Z) = 3Z[\delta(n+2)] + 2Z[\delta(n)] + Z[\delta(n-1)] - Z[\delta(n-2)]$$

Using the time shifting property,

$$X(Z) = 3Z^2 + 2 + Z^{-1} - Z^{-2}$$

ROC: Entire Z-plane except at $Z = 0$ and $Z = \infty$.

4. Find the Z-transform of $x(n) = n\,u(n)$

Solution:

Given $x(n) = n\,u(n)$

$$\text{w.k.t, } Z[u(n)] = \frac{1}{1 - Z^{-1}} = \frac{Z}{Z - 1};\ \text{ROC:}|Z| < 1$$

Using the multiplication by n property

$$Z[n\,u(n)] = -Z\frac{d}{dZ}\left[\frac{Z}{Z - 1}\right]$$

$$= -Z\left[\frac{(Z - 1)(1) - Z(1)}{(Z - 1)^2}\right]$$

$$= -Z\left[\frac{Z - 1 - Z}{(Z - 1)^2}\right]$$

$$X(Z) = \frac{Z}{(Z - 1)^2};\ \text{ROC:}|Z| > 1$$

5. Find the Z-transform of the sequence $x(n) = a^n \cos\frac{n\pi}{2} u(n)$

Solution:

Given $x(n) = a^n \cos\frac{n\pi}{2} u(n)$

w.k.t, $Z[\cos\omega n\, u(n)] = \frac{Z^2 - Z\cos\omega}{Z^2 - 2Z\cos\omega + 1}$

For $\omega = \frac{\pi}{2}$

$$Z\left[\cos\frac{n\pi}{2} u(n)\right] = \frac{Z^2}{Z^2 + 1}$$

Using the scaling property,

$$Z[a^n x(n)] = X(a^{-1} Z)$$

$$Z\left[a^n \cos\frac{n\pi}{2} u(n)\right] = \frac{(a^{-1}Z)^2}{(a^{-1}Z)^2 + 1}$$

$$= \frac{a^{-2}Z^2}{a^{-2}Z^2 + 1}$$

$$= \frac{Z^2}{Z^2 + a^2}$$

6. Find the Z-transform of $x(n) = a^n \sin \omega n\, u(n)$

Solution:

$$\text{w.k.t, } Z[\sin \omega n\, u(n)] = \frac{Z \sin \omega}{Z^2 - 2Z \cos \omega + 1}$$

Using the scaling property

$$Z[a^n x(n)] = X(a^{-1} Z)$$

$$Z[a^n \sin \omega n\, u(n)] = \frac{a^{-1} Z \sin \omega}{\left(a^{-1} Z\right)^2 - 2\left(a^{-1} Z\right) \cos \omega + 1}$$

$$= \frac{a Z \sin \omega}{Z^2 - 2a Z \cos \omega + a^2}.$$

DISCRETE-TIME FOURIER TRANSFORM (DTFT)

A continuous time signal can be represented in the frequency domain using laplace transform (or) continuous time fourier transform. Similarly a discrete time signal can be represented in the frequency domain using z-transform (or) discrete time fourier transform.

The fourier transform of discrete time signal is called discrete-time fourier transform (DTFT).

If $x(n)$ is the given discrete time sequence, then $X(\omega)$ (or) $X\left(e^{j\omega}\right)$ is the discrete time fourier transform of $x(n)$.

The DTFT of x(n) is defined as

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

The inverse DTFT of $X(\omega)$ is defined as

$$F^{-1}[X(\omega)] = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega$$

Existence of DTFT

* The fourier transform exists for a discrete time sequence $x(n)$ if and only if the sequence is absolutely summable, i.e., the sequence has to satisfy the condition.

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

* The DTFT does not exist for the sequences that are growing exponentially (ex. $a^n u(n)$, $a > 1$). Since they are not absolutely summable.

Comparison of fourier transform of a discrete time signal and continuous time/analog signal.

	Continuous time fourier transform	Discrete time fourier transform
(i)	The fourier transform of analog signals consists of a spectrum with a frequency range $-\infty$ to ∞.	The fourier of discrete time signal is unique in the frequency range $-\pi$ to π (or 0 to 2π). Also fourier transform of discrete time signals are periodic with period 2π.
(ii)	The analog signals are continuous, the fourier transform of analog signals involves discrete.	The fourier transform of discrete time signals involves summation because the signals are

1. Find the DTFT of $x(n) = \delta(n)$.

Solution:

Given $x(n) = \delta(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(\omega) = F[\delta(n)] = \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega n}$$

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega n}\Big|_{n=0} = 1$$

$F[\delta(n)] = 1$.

2. Find the DTFT of $x(n) = u(n)$.

Solution:

Given $x(n) = u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(\omega) = F[u(n)] = \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\omega n}$$

$$u(n) = \begin{cases} 1 & \text{for } n > 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$X(\omega) = \sum_{n=0}^{\infty} (1)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(e^{-j\omega}\right)^n = \frac{1}{1 - e^{-j\omega}} \qquad \left[\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\right]$$

$$\mathbf{F[u(n)] = \frac{1}{1 - e^{-j\omega}}}.$$

(i) Given $x(n) = \{1, 2, -1, 3\}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

The sequence values are $x(0) = 1$

$x(1) = 2, x(2) = -1, x(3) = 3$

$$X(\omega) = x(0) + x(1)e^{-j\omega} + x(2)e^{-2j\omega} + x(3)e^{-3j\omega}$$

$$X(\omega) = 1 + 2e^{-j\omega} - e^{-2j\omega} + 3e^{-3j\omega}$$

$$x(n) = (0.5)^n\, u(n) + 2^n u(-n-1)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left\{ (0.5)^n\ u(n) + 2^n\ u(-n\ -\ 1) \right\} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (0.5)^n\ e^{-j\omega n} + \sum_{n=1}^{\infty} \left(2^{-1}\ e^{-j\omega} \right)^n$$

$$= \sum_{n=0}^{\infty} \left(0.5\ e^{-j\omega} \right)^n + \sum_{n=1}^{\infty} \left(2^{-1}\ e^{-j\omega} \right)^n$$

$$= \frac{1}{1 - 0.5 e^{-j\omega}} + \frac{2^{-1} e^{-j\omega}}{1 - 2^{-1} e^{-j\omega}}$$

$$X(\omega) = \frac{1}{1 - 0.5 e^{-j\omega}} - \frac{1}{1 - 2 e^{-j\omega}}$$

(i) Given, $x(n) = \frac{1}{3}\left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{4}\right)^n\right]u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{3}\left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{4}\right)^n\right] u(n)e^{-j\omega n}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n e^{-j\omega n} + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n e^{-j\omega n}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j\omega}\right)^{n} + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^{n}$$

$$= \frac{1}{3}\frac{1}{1-\frac{1}{3}e^{-j\omega}} + \frac{1}{3}\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{1}{3}\left[\frac{1}{1-\frac{1}{3}e^{-j\omega}} + \frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$

$$= \frac{1}{3}\left[\frac{1-\frac{1}{4}e^{-j\omega}+1-\frac{1}{3}e^{-j\omega}}{\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]$$

$$X(\omega) = \frac{1}{3}\left[\frac{2-\frac{7}{12}e^{-j\omega}}{\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]$$

1. Linearity property

The Linearity property of DTFT states that

If $F[x_1(n)] = X_1(\omega)$ and $F[x_2(n)] = X_2(\omega)$

then $F[ax_1(n) + bx_2(n)] = aX_1(\omega) + bX_2(\omega)$

Proof:

$$F[ax_1(n) + bx_2(n)] = \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} ax_1(n)e^{-j\omega n} + \sum_{n=-\infty}^{\infty} bx_2(n)e^{-j\omega n}$$

$$= aX_1(\omega) + bX_2(\omega)$$

2. Periodicity property

The periodicity property of DTFT states that the DTFT $X(\omega)$ is periodic in ω with period 2π.

i.e., $X(\omega + 2n\pi) = X(\omega)$ for any integer n.

3. Time shifting property

The time shifting property of DTFT states that

If $F[x(n)] = X(\omega)$

then $F[x(n - n_0)] = e^{-j\omega n_0} X(\omega)$ where n_0 is an integer

Proof:

$$F[x(n - n_0)] = \sum_{n=-\infty}^{\infty} x(n - n_0) e^{-j\omega n}$$

Let $n - n_0 = K$, $n = K + n_0$

$$F[x(n - n_0)] = \sum_{K=-\infty}^{\infty} x(K) e^{-j\omega K}$$

$$= e^{-j\omega n_0} \sum_{K=-\infty}^{\infty} x(K) e^{-j\omega K}$$

$$= e^{-j\omega n_0} X(\omega).$$

The time shifting of a signal by n_0 units does not change its amplitude spectrum but the phase spectrum is changed by $-\omega n_0$.

4. Frequency shifting property

The frequency shifting property of DTFT states that

If $F[x(n)] = X(\omega)$

then $F[x(n)\, e^{j\omega_0 n}] = X(\omega - \omega_0)$

Proof:

$$F[x(n)\, e^{j\omega_0 n}] = \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega_0 n}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega - \omega_0)n} = X(\omega - \omega_0)$$

This property is the dual of the time shifting property.

5. Time Reversal property

The time reversal property of DTFT states that

If $F[x(n)] = X(\omega)$

then $F[x(-n)] = X(-\omega)$

Proof:

$$F[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\omega)n}$$

$$= X(-\omega)$$

i.e., folding in the time domain corresponds to the folding in the frequency domain.

6. Differentiation in the frequency domain property

The differentiation in the frequency domain property of DTFT states that

If $F[x(n)] = X(\omega)$

then $F[nx(n)] = j\frac{d}{d\omega}[X(\omega)]$

Proof:

$$F[x(n)] = [X(\omega)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Differentiating both sides w.r.t ω, we get

$$\frac{d}{d\omega}[X(\omega)] = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-jn)e^{-j\omega n}$$

$$= -j \sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}$$

$$j\frac{d}{d\omega}[X(\omega)] = \sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}$$

$$\therefore \; F[nx(n)] = j\frac{d}{d\omega}[X(\omega)] .$$

7. Time convolution property

The time convolution properyt of DTFT states that

If $F[x_1(n)] = X_1(\omega)$ and $F[x_2(n)] = X_2(\omega)$

then $F[x_1(n) * x_2(n)] = X_1(\omega)\, X_2(\omega)$

Proof:

$$x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K)$$

$$\therefore \quad F[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K) \right\} e^{-j\omega n}$$

Interchanging the order of summations, we get

$$F[x_1(n) * x_2(n)] = \sum_{K=-\infty}^{\infty} x_1(K) \sum_{n=-\infty}^{\infty} x_2(n-K) e^{-j\omega n}$$

Put $n - K = p$ in the second summation

$$\therefore n = p + K$$

$$F[x_1(n) * x_2(n)] = \sum_{K=-\infty}^{\infty} x_1(K) \sum_{p=-\infty}^{\infty} x_2(p) e^{-j\omega(p+K)}$$

$$= \sum_{K=-\infty}^{\infty} x_1(K) e^{-j\omega K} \sum_{p=-\infty}^{\infty} x_2(p) e^{-j\omega p}$$

$$= X_1(\omega) X_2(\omega)$$

The convolution of the signals in the time domain is equal to multiplying their spectra in the frequency domain.

Find the DTFT of $x(n) = \left(\frac{1}{2}\right)^n u(n)$ and plot its spectrum.

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^n$$

$$X(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$= \frac{1}{1 - \frac{1}{2}\cos\omega + \frac{j}{2}\sin\omega}$$

The magnitude spectrum is

$$|X(\omega)| = \frac{1}{\sqrt{\left(1 - \frac{1}{2}\cos\omega\right)^2 + \frac{1}{4}\sin^2\omega}}$$

$$= \frac{1}{\sqrt{1 + \frac{1}{4}\cos^2\omega - \cos\omega + \frac{1}{4}\sin^2\omega}}$$

$$= \frac{1}{\sqrt{1 + \frac{1}{4} - \cos\omega}}$$

$$= \frac{1}{\sqrt{\frac{5}{4} - \cos\omega}}$$

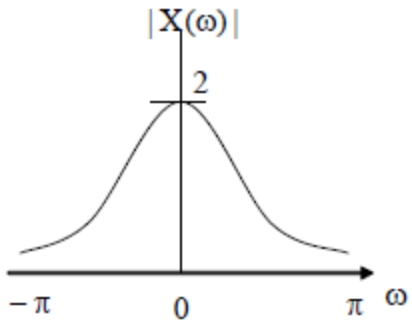
$$|X(\omega)| = \frac{2}{\sqrt{5 - 4\cos\omega}}$$

The phase spectrum is,

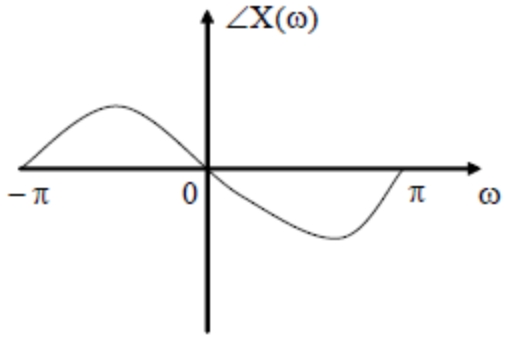
$$\angle X(\omega) = -\tan^{-1}\left[\frac{\frac{1}{2}\sin\omega}{1-\frac{1}{2}\cos\omega}\right]$$

ω	$\lvert X(\omega)\rvert$	$\angle X(\omega)$
$-\pi$	0.67	0°
$-\frac{\pi}{2}$	0.89	26.6°
$-\frac{\pi}{4}$	1.36	28.7°
0	2	0°
$\frac{\pi}{4}$	1.36	-28.7°
$\frac{\pi}{2}$	0.89	-26.6°
π	0.67	0°

Magnitude spectrum



Phase spectrum



Find the inverse fourier transform of

$$X(\omega) = \begin{cases} 1, & \frac{\pi}{2} \le |\omega| \le \frac{3\pi}{2} \\ 0, & \frac{3\pi}{2} \le |\omega| \le \pi \end{cases}$$

Solution:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-3\pi/2}^{-\pi/2} e^{j\omega n}\, d\omega + \int_{\pi/2}^{3\pi/2} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)_{-3\pi/2}^{-\pi/2} + \left(\frac{e^{j\omega n}}{jn}\right)_{\pi/2}^{3\pi/2}\right]$$

$$= \frac{1}{2\pi jn}\left[e^{-jn\pi/2} - e^{-jn3\pi/2} + e^{jn3\pi/2} - e^{jn\pi/2}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{jn3\pi/2} - e^{-jn3\pi/2}}{2j} - \left(\frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2j}\right)\right]$$

$$x(n) = \frac{1}{\pi n}\left[\sin n\,3\pi/2 - \sin n\,\pi/2\right].$$

Find the inverse fourier transform of

(i) $X(\omega) = e^{-j\omega}(1 + \cos\omega)$ **(ii)** $X(\omega) = 1 + e^{-j\omega} + 2e^{-4j\omega}$

Solution:

(i) Given $X(\omega) = e^{-j\omega}(1 + \cos\omega)$

$$= e^{-j\omega}\left[\frac{1 + e^{j\omega} + e^{-j\omega}}{2}\right]$$

w.k.t, $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$

$$= e^{-j\omega} + 0.5 + 0.5e^{-2j\omega}$$

$$= 0.5 + e^{-j\omega} + 0.5e^{-2j\omega} \quad \text{---------- (1)}$$

$$= x(0) + x(1)e^{-j\omega} + x(2)e^{-2j\omega} \quad \text{---------- (2)}$$

Comparing the equations (1) & (2)

Therefore, $x(0) = 0.5$, $x(1) = 1$ $x(2) = 0.5$

(ii) Given $X(\omega) = 1 + e^{-j\omega} + 2e^{-4j\omega}$

w.k.t, $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$

Interval, $0 \leq n \leq 4$

$$X(\omega) = x(0) + x(1)e^{-j\omega} + x(2)e^{-2j\omega} + x(3)e^{-3j\omega} + x(4)e^{-4j\omega}$$

Comparing the above equation with the given problem we get

$$x(0) = 1, x(1) = 1 , x(2) = 0, x(3) = 0, x(4) = 2$$

$$x(n) = \left\{\underset{\uparrow}{1}, 1, 0, 0, 2\right\}.$$

SAMPLING OF CONTINUOUS TIME (CT) SIGNALS

Sampling of the signal is the fundamental operation in signal processing. The process of converting a continous time signal into a discrete time signal is called sampling. After sampling, the signal is defined at discrete instants of time and the time interval between two successive sampling instants is called sampling period (or) sampling interval. The sampling rate is the inverse of the sampling period. The sampling rate must be kept sufficiently high so that the original signal can be reconstructed from its samples.

Let us discuss the sampling theorem. The statement of sampling theorem can be given in two parts as:

(i) A band-limited signal of finite energy, which has no frequency component higher than f_m Hz, is completely described by its sample values at uniform intervals less than or equal to $\frac{1}{2f_m}$ second.

(ii) A band-limited signal of finite energy, which has no frequency components higher than f_m Hz, may be completely recovered from the knowledge of its samples taken at the rate of $2f_m$ samples per second.

Combining the two parts, the sampling theorem may be state as:

"A continuous time signal may be completely represented in its samples and recovered back if the sampling frequency is $f_s \geq 2f_m$. Here f_s is the sampling frequency and f_m is the maximum frequency present in the signal".

Proof of sampling theorem

Let $x(t)$ be the continuous bandlimited signal which has no frequency components higher than f_m Hz.

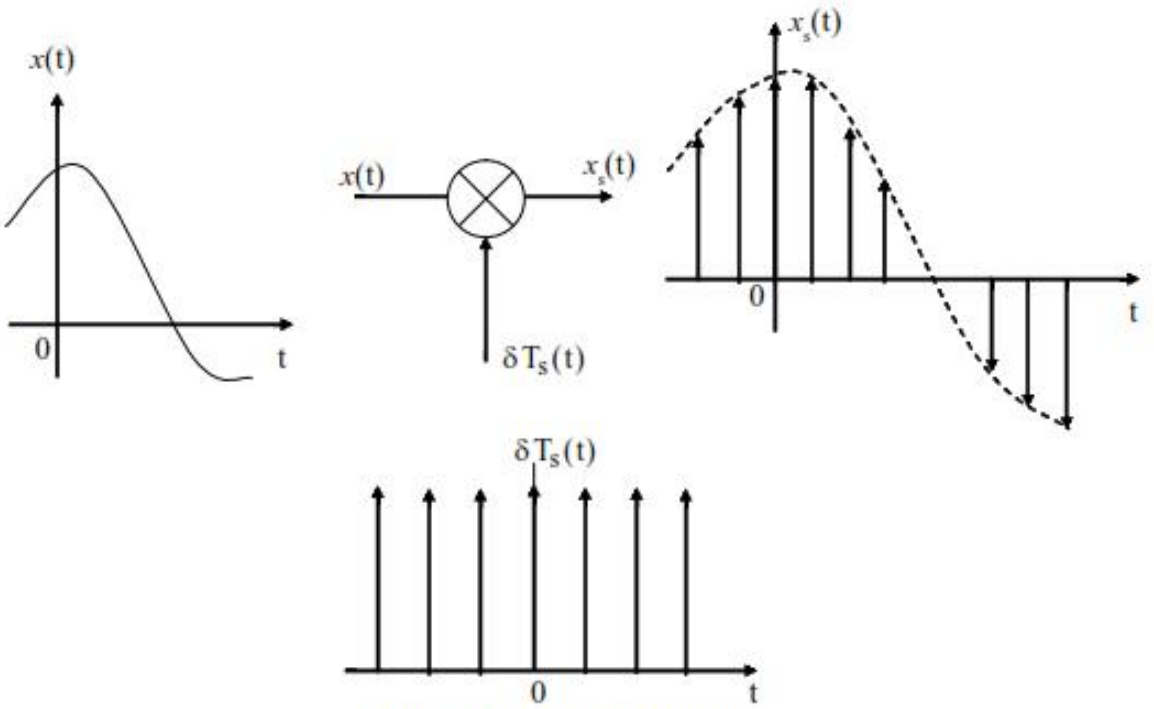


Figure: Sampling operation

Let this signal be sampled by a sampling function given as

$$\delta T_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The sampling function is the train of impulses with T_s as distance between successive impulses.

Let $x(nT_s)$ be the instantaneous amplitude of signal $x(t)$ at instant $t = T_s$. The sampled version of $x(t)$ is given as multiplication of $x(nT_s)$ and $\delta T_s(t)$.

$$\text{i.e.,} \quad x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \qquad \text{---------- (1)}$$

Now taking the fourier transform for this sampled signal.

$$X_s(f) = F.T[x_s(t)]$$

The exponential form of fourier series of $\delta T_s(t)$ is,

$$\delta T_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t}$$

where $$C_n = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) e^{-jn\omega_s t}\, dt$$

$$= \frac{1}{T_s}$$

$$\delta T_s(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$X_s(f) = F.T[x_s(t)] = \frac{1}{T_s} \sum_{n=-\alpha}^{\alpha} F.T\left[x(nT_s) e^{jn\omega_s t} \right]$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \qquad [\because \text{ from frequency shifting property}]$$

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad \text{----------(2)}$$

Here, f_s is the sampling rate which is given as $f_s = \frac{1}{T_s}$ and $X(f - nf_s) = x(f)$ at $nf_s = 0, \pm f_s, \pm 2f_s$ .....

Hence, the same spectrum X(f) appears at $f = 0, \pm f_s, \pm 2f_s$ .....

$X_s(f)$ in equation (2) can be expanded as

$$X_s(f) = f_s X(f) + f_s X(f + f_s) + f_s X(f + 2f_s)$$

It can be also written as

$$X_s(f) = f_s X(f) + \sum_{n=-\infty}^{\infty} f_s X(f - nf_s)$$

$$X_s(f) = \frac{1}{f_s} X_s(f) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} X(f - nf_s)$$

We know that $x(t)$ is strictly bandlimited to f_m Hz.

$$\therefore \quad X(f) = \frac{1}{f_s} X_s(f) \qquad \text{for } -f_m \leq f \leq f_m$$

Applying fourier transform to Equation (1) yields

$$F.T[x_s(t)] = F.T\left[\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)\right]$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) F.T[\delta(t - nT_s)]$$

We have $F.T[\delta(t - nT_s)] = \int_{-\infty}^{\infty} \delta(t - nT_s) e^{-j\omega t} dt$

$$= e^{-j\omega nT_s} = e^{-j2\pi fn.T_s}$$

$$\therefore \quad X_s(f) = F.T[x_s(t)] = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi fnT_s}$$

Now putting the $X_s(f)$ in $X(f)$, we get

$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi fnT_s}$$

Since $f_s = 2f_m$; $T_s = \dfrac{1}{2f_m}$

$$X(f) = \frac{1}{2f_m} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) e^{-j\pi fn/f_m}$$

$x(t)$ may be recovered from X(f) by taking Inverse fourier transform of equation

$$x(t) = \text{I.F.T}\left\{\frac{1}{2f_m} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) e^{-j\pi fn/f_m}\right\}$$

This equation indicates that $x(t)$ is represented completely by its samples $x\left(\dfrac{n}{2f_m}\right)$ for $-\infty < n < \infty$.

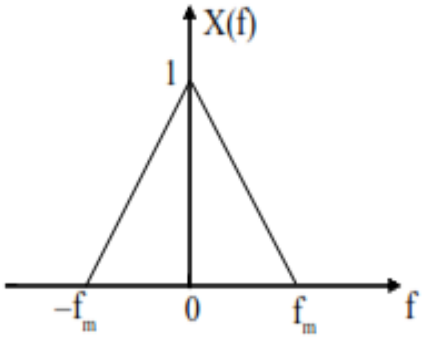


Figure : Frequency Spectrum of $x(t)$

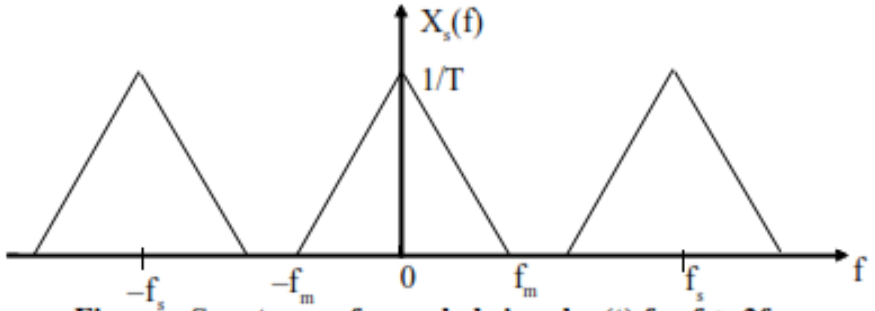


Figure : Spectrum of sampled signal $x_s(t)$ for $f_s > 2f_m$

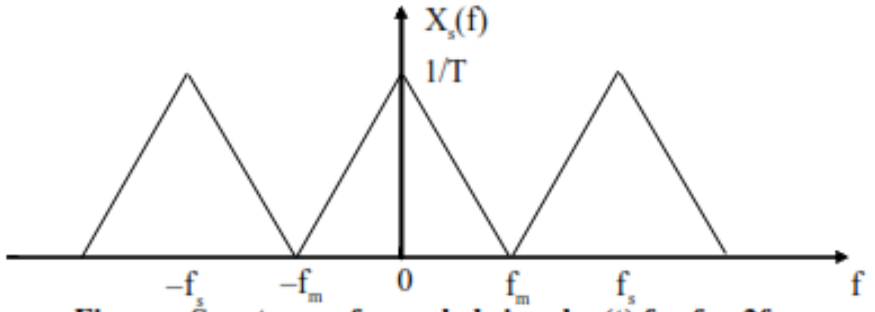


Figure : Spectrum of sampled signal $x_s(t)$ for $f_s = 2f_m$

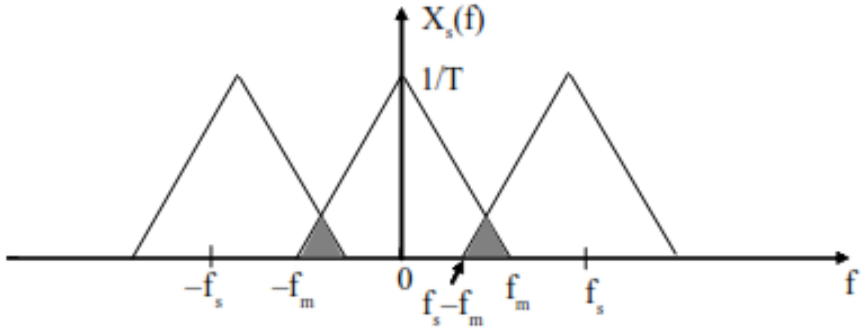


Figure : Spectrum of sampled signal $x_s(t)$ for $f_s < 2f_m$

From above figures, we observe that

(i) $X_s(f)$ is a repetitive version of X(f) with X(f) repeating itself at regular intervals of f_s.

(ii) When $f_s > 2f_m$, the spectral replicates have a larger separation between them known as guardband which makes the successive cycles of $X_s(f)$ are not overlapping each other. Hence in this case there is no problem in recovering the original spectrum X(f).

(iii) When $f_s = 2f_m$, the successive cycle of Xs(f) are not overlapping each other, but they are touching each other. In this case, the original spectrum X(f) can be recovered from the sampled spectrum $X_s(f)$ using a low pass filter with cut-off frequency f_m.

(iv) When $f_s < 2f_m$, the successive cycles of the sampled spectrum will overlap each other and there is distortion and X(f) cannot be recovered from $X_s(f)$. This type of distortion is called aliasing. Aliasing can be avoided if $f_s > 2f_m$.

Nyquist rate and Nyquist Interval

Nyquist rate of sampling is the minimum sampling rate at which a signal can be sampled and still be reconstructed from its samples without any distortion.

When the sampling rate becomes exactly equal to $2f_m$ samples per second, then it is called Nyquist rate. It is also called the minimum sampling rate.

It is given by

$$\mathbf{f_s = 2f_m}$$

A signal sampled at greater than Nyquist rate is said to be oversampled and a signal sampled at less than its Nyquist rate is said to be undersampled.

Nyquist interval is the time interval between any two adjacent samples when sampling rate is Nyquist rate.

The maximum sampling interval is called Nyquist intervals (T_s). It is given by

$$\mathbf{T_s = \frac{1}{2f_m}\ seconds}$$

Effects of under sampling-Aliasing

If the signal $x(t)$ is not strictly bandlimited and if the sampling frequency f_s is less than $2f_m$, the adjacent spectrums overlapped then an error called aliasing is observed.

A high frequency in the spectrum of the original signal $x(t)$, taking on the identity of lower frequency in the spectrum of the sampled signal $x_s(t)$ is called as aliasing (or) fold over error (or) frequency folding effect.

Effect of Aliasing

Due to aliasing some of the information contained in the original signal $x(t)$ is lost in the process of sampling.

How to Eliminate Aliasing?

(i) Use a bandlimiting low pass filter (LPF) and pass the signal $x(t)$ through it before sampling. The filter has a cutoff frequency at $f_c = f_m$, therefore it will strictly bandlimit the signal $x(t)$ before sampling. This filter is also called as antiliasing filter (or) prealiasing filter.

(ii) Use sampling frequency (f_s) is greater than $2f_m$.

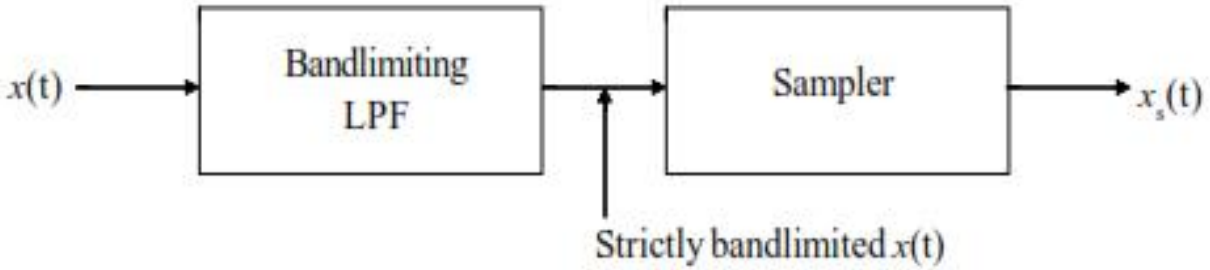


Figure : Use of Bandlimiting filter

4.1.2 Reconstruction of signal from samples

The input signal $x(t)$ can be reconstructed by taking Inverse fourier transform of X(f).

$$x(t) = \frac{1}{2f_m} \int_{-f_m}^{f_m} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) e^{-j\pi fn/f_m} e^{j2\pi ft}\, df$$

Interchanging the order of summation and integration, we get

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{1}{2f_m} \int_{-f_m}^{f_m} e^{j2\pi f}\left(t - \frac{n}{2f_m}\right) df$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{1}{2f_m\left[j2\pi t - \frac{j2\pi n}{2f_m}\right]}\left[e^{j2\pi f_m\left(t-\frac{n}{2f_m}\right)} - e^{-j2\pi f_m\left(t-\frac{n}{2f_m}\right)}\right]$$

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{1}{2f_m\left[\frac{j2\pi t.2f_m - j2\pi n}{2f_m}\right]} \times \left[e^{j2\pi f_m\left(t-\frac{n}{2f_m}\right)} - e^{-j2\pi f_m\left(t-\frac{n}{2f_m}\right)}\right]$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \cdot \frac{e^{j2\pi f_m\left(t-\frac{n}{2f_m}\right)} - e^{-j2\pi f_m\left(t-\frac{n}{2f_m}\right)}}{j2[2\pi f_m t - \pi n]}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{\sin(2\pi f_m t - n\pi)}{(2\pi f_m t - n\pi)}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{\sin \pi(2f_m t - n)}{\pi(2f_m t - n\pi)}$$

Since $\sin c\theta = \dfrac{\sin \pi\theta}{\pi\theta}$

$$\therefore\ x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right)\sin c(2f_m t - n)$$

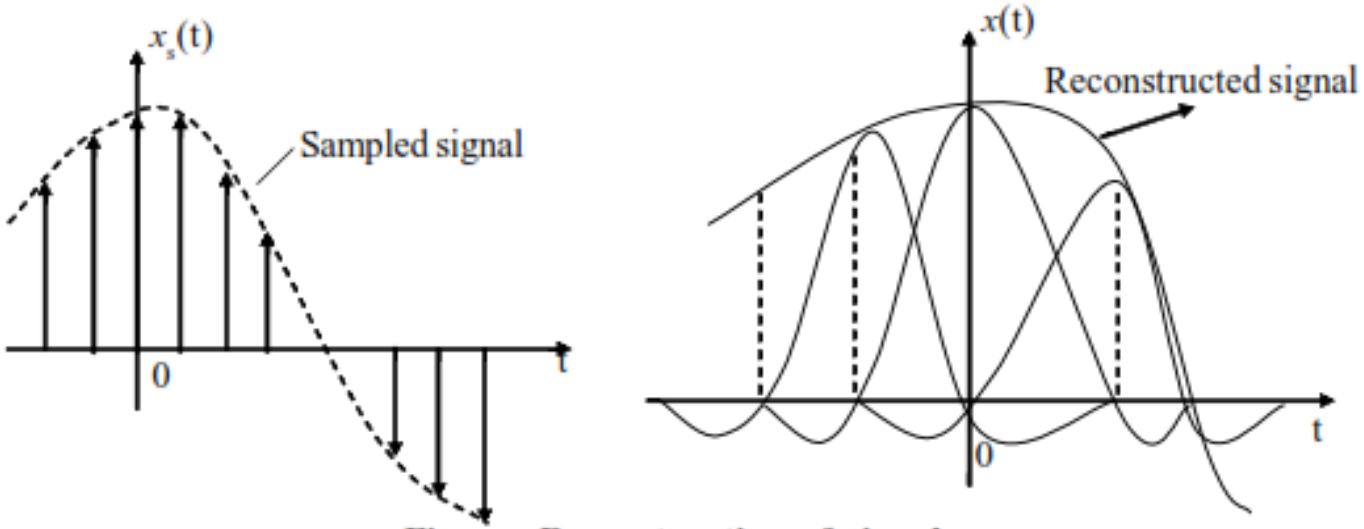


Figure: Reconstruction of signal

Hence this is the interpolation formula to reconstruct $x(t)$ from its samples $_x(nT_s)$. Therefore from all above it is clear that the signal may be completely represented into and recovered from its samples if the spacing between the successive samples is $1/2f_m$ seconds i.e., $f_s = 2f_m$ samples per second.